# Complex Variables Workshop

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**Abstract**

This document contains what I intend to cover in the preparatory workshop for the NYU Courant January 2026 complex variables written exams. An emphasis is placed on using key ideas and theorems to solve problems. Thus, most proofs of the main theorems are omitted, unless the proof is particularly enlightening or may appear on the exam. The exact topics covered on each day of the workshop may change depending on how fast/slow we can work through the material.
The date this document was last updated is listed in the top-right corner of each page. The references I used to structure these notes are Brown & Churchill's *Complex Variables and Applications* [1] and Stein & Shakarchi's *Complex Analysis* [2].

# Contents

# Day 1: Review & Differentiation

## 1.1 Notation and Review

We will define the *complex plane* by

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\} \cong \mathbb{R}^2,$$

where $i^2 = -1$. We write $z \in \mathbb{C}$ to mean that there exists $x, y \in \mathbb{R}$ such that $z = x + iy$. We write $x = \text{Re}z$ and $y = \text{Im}z$, saying that $x$ and $y$ are the *real and imaginary parts* of $z$, respectively. Addition, multiplication, and division are defined as one would expect,

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2)\\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\\ \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}.\end{aligned}$$

We will denote the *modulus* of a complex number $z$ by the usual Euclidean norm

$$|z| = \sqrt{x^2 + y^2}$$

and *complex conjugation* by

$$\overline{z} = x - iy.$$

It is often convenient to express a complex number in *polar form*, $z = re^{i\theta}$, where $r = |z|$, $\theta = \arg(z)$, and

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

**Example 1.1:** Find all cube roots of $1 + i$.

*Solution.* Write $z^3 = 1 + i = \sqrt{2}e^{i(\pi/4+2\pi k)}$, to find that

$$z = 2^{1/6}e^{\pi/12},\ 2^{1/6}e^{3\pi/4},\ 2^{1/6}e^{17\pi/12}.$$

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We define the *principal argument* of $z$, denoted by $\text{Arg}(z)$, to be the unique value in $(-\pi, \pi]$ such that

$$\arg(z) = \text{Arg}(z) + 2\pi n \quad \forall n \in \mathbb{Z}.$$

While it is true that $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$, this is not necessarily true for Arg (you should find a counterexample).

Note that the definition of $e^{i\theta}$ immediately gives a definition of the complex exponential function $e^z$ as

$$e^z = e^xe^{iy} = e^x(\cos(y) + i\sin(y)).$$

In these notes, any non-empty, open, connected set $D$ is called a *domain.*

## 1.2 Analytic Functions

### 1.2.1 Derivatives of Complex Functions

A function $f\colon S \subseteq \mathbb{C} \to \mathbb{C}$ can be decomposed into real component functions $u, v$ via

$$f(z) = f(x+iy) = u(x,y) + iv(x,y).$$

The notions of limits and continuity are defined as one would on a metric space, so we will move right on to discussing differentiation of complex functions.

The derivative of $f$ at a point $z_0 \in \mathbb{C}$ is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Taking $\delta z = z - z_0$, this is equivalent to

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}.$$

The classical example of a complex function that is not differentiable is the following:

**Example 1.2:** The conjugation map $z \mapsto \overline{z}$ is not differentiable at the origin.

*Solution:* Assume instead $f(z) = \overline{z}$ were to be differentiable at $z = 0$. Then the limit of the difference quotient exists on any path. Comparing the paths $(\delta x, 0)$ and $(0, \delta y)$, we find a contradiction.

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We note that linearity, the Leibnitz rule, the quotient rule, and the chain rule are all true for differentiable complex functions.

It is also important to note that complex differentiation is not the same as differentiation in $\mathbb{R}^2$. Namely, differentiation in $\mathbb{R}^2$ is like linearization, while complex differentiation is about angle preservation.

**Exercise 1.1:** Consider the maps $f(x,y) = x$ and $g(z) = \mathrm{Re}(z)$. Show that $f$ is differentiable as a map from $\mathbb{R}^2 \to \mathbb{R}^2$ but $g$ is not complex differentiable.

### 1.2.2 Cauchy-Riemann Equations

Perhaps the most important system of partial differential equations (PDEs) in complex analysis are the *Cauchy-Riemann equations*:

**Theorem 1.1 (C-R Equations)**

Suppose $f = u + iv$ is differentiable at a point $z_0 = z_0 + iy_0$. Then the first order partial derivatives of $u, v$ must exist at $(x_0, y_0)$, and they satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Moreover,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

The idea in deriving the C-R equations is to again look at the limit definition of the derivative along the paths $(\delta x, 0)$ and $(0, \delta y)$.

It is important to note that the converse statement is generally not true without additional assumptions on $u$ and $v$.

**Theorem 1.2**

Suppose that $f = u + iv$ is defined in some neighbourhood of a point $z_0 = x_0 + iy_0$, and that $u, v$ are continuously differentiable in the neighbourhood and satisfy the C-R equations at $(x_0, y_0)$. Then $f'(z_0)$ exists, with

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

This result is the first instance where we see the importance of the local existence of the derivative. We will soon come back to this when we discuss analyticity and integration.

We quickly note that, in polar coordinates, the C-R equations become

$$ru_r = u_\theta, \quad u_\theta = -rv_r.$$

### 1.2.3 Analyticity

We say that $f$ is *holomorphic* (or *analytic*) at a point $z_0$ if there exists a neighbourhood of $z_0$ such that $f$ is differentiable at each point in that neighbourhood. If $f$ is defined on an open set, it is called holomorphic on that set if it is differentiable at each point. A function is called *entire* if it is defined and holomorphic on all of $\mathbb{C}$. Given a domain $D$, the set of holomorphic functions on $D$ is denoted by $\mathcal{H}(D)$.

We note that every holomorphic function on a domain $D$ satisfies the C-R equations, and the previous theorems can also be used to prove a function is holomorphic. Another important fact is that if $f$ is holomorphic in a domain $D$, then the component functions $u$ and $v$ are *harmonic*:

$$\Delta u, \Delta v = 0, \quad \text{where } \Delta = \partial_{xx} + \partial_{yy}.$$

In fact, more can be said, but we first need the following definition. If $u, v$ are harmonic in $D$ and satisfy the C-R equations, then $v$ is called a *harmonic conjugate* of $u$.

**Theorem 1.3**

A function $f = u + iv$ is holomorphic in a domain $D$ if and only if $v$ is a harmonic conjugate of $u$.

**Example 1.3:** Let $u(x,y) = x^2 - y^2$. Find a harmonic conjugate of $u$.

*Solution:* First, note that it is easy to see that $u$ is harmonic. If $v$ is a harmonic conjugate of $u$, then

$$v_x = -u_y = 2y, \quad v_y = u_x = 2x.$$

Integrating, we see that $v(x,y) = 2xy$ works.

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**Exercise 1.2:** Let $u(x,y) = e^x(x\cos(y) - y\sin(y))$.
(a) Show that $u$ is harmonic on $\mathbb{R}^2$.
(b) Find a harmonic function $v$ such that $f = u + iv$ is holomorphic on $\mathbb{C}$.
(c) Express $f(z)$ explicitly as a function of $z$, where $z = x + iy$.

**Example 1.4 (January 2017, Problem 1):** Define the function

$$f(z) = \begin{cases} \exp\left(-\frac{1}{z^4}\right) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

1. Show that $f$ satisfies the Cauchy-Riemann equations on $\mathbb{C}$.
2. Is $f$ entire?

*Solution:*

(a) First note that, away from $z = 0$, $f$ is the composition of holomorphic functions. Hence, $f$ satisfies the C-R equations if $z \neq 0$.

Since we are considering the point $(x,y) = (0,0)$, it suffices to compute the partial derivatives along $x = 0$ and $y = 0$. Decomposing $f$ as $f = u + iv$, it is easy to see that along the axes we have $v \equiv 0$. Meanwhile, $u(x,0) = \exp(-1/x^4)$, $u(0,y) = \exp(-1/y^4)$. It is a straightforward computation using L'Hopital to show that $u_x(0,0) = u_y(0,0) = 0$.

(b) We show $f$ is not continuous at the origin, which proves $f$ is not holomorphic there. To see this, we compute the limit of $f$ as $z \to 0$ along the line $x = y$. Writing $z = re^{i\pi/4}$, this becomes

$$\lim_{r\to 0} f(re^{i\pi/4}) = \lim_{r\to 0} \exp\left(\frac{1}{r^4}\right) = \infty \neq 0 = f(0).$$

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**Example 1.5:** Let $f(z) = u + iv$ be a holomorphic function on some domain $D \subseteq \mathbb{C}$. Assume there exists $\Phi \in C^1(\mathbb{R}^2 \to \mathbb{R})$ such that

$$\Phi(u(x, y), v(x, y)) = 0 \quad \forall z = x + iy \in D.$$

Assume further that $\nabla\Phi \neq 0$ on $f(D)$. Show that $f$ is a constant.

*Solution:* Differentiating $\Phi(u, v) = 0$ with respect to $x$ and $y$ and using the C-R equations, we find that

$$\frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial\Phi}{\partial v}\frac{\partial u}{\partial x} - \frac{\partial\Phi}{\partial u}\frac{\partial v}{\partial x} = 0.$$

This is a system of equations where the determinant of the coefficient matrix is $-(\Phi_u^2 + \Phi_v^2) \neq 0$ on $f(D)$. Thus, this system only has the trivial solution for all $z \in D$, so that

$$f'(z) = u_x + iv_x = 0 \quad \forall z \in D.$$

Hence, $f$ is a constant.

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**Example 1.6:** Let $f : D \to \mathbb{C}$ be holomorphic, $D$ a domain. Suppose there exists a constant $\alpha \in \mathbb{R}$ such that

$$\arg(f'(z)) = \alpha \quad \forall z \in D \backslash \{f'(z) = 0\}.$$

Show that $f$ is of the form $f(z) = a + bz$ for some $a, b \in \mathbb{C}$.

*Solution:* We will use that derivatives of holomorphic functions are holomorphic (we will discuss this more later). Since $f'$ lives on the ray at the angle $\alpha$, we can write

$$f'(z) = |f'(z)|e^{i\alpha}.$$

Define $g(z) = e^{-i\alpha}f(z)$. Since $f$ is holomorphic on $D$, so is $g$. Moreover, $g'(z) = |f'(z)|$, so $g'$ is a real-valued holomorphic function. In particular, the C-R equations guarantee that $g'$ is a constant. Integrating (which we will also discuss later), we see that $g(z)$ is affine.

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**Example 1.7 (September 2013, Problem 2):** Let $z = x + iy$, $f = f(z) = u + iv$. Assume $D$ is a domain in $\mathbb{C}$, $f \in C^2(D)$. Denote

$$Df = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}.$$

Suppose for every $z \in D$ that

$$Df(z)^T Df(z) = \lambda(z) I,$$

where $I$ is the $2 \times 2$ identity matrix. Show that either $f$ or $\overline{f}$ is holomorphic in $D$.

*Solution:* Since $Df^T Df = \lambda I$, we have that

$$\begin{aligned}(\partial_x u)^2 + (\partial_y v)^2 = \lambda = (\partial_y u)^2 + (\partial_y v)^2,\\ (\partial_x u)(\partial_y u) + (\partial_x v)(\partial_y v) = 0.\end{aligned}$$

Note that the second condition is saying that the columns of $Df$ are orthogonal, while the first says that they have the same length (and $\lambda$ is a non-negative function). Therefore, each column is precisely a rotation of the other by $\pi/2$ or $-\pi/2$.

Let $f_1$ and $f_2$ denote the first and second column of $Df$, respectively. If $f_2$ is a rotation by $\pi/2$ of $f_1$, then

$$\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} -v_x \\ u_x \end{pmatrix}.$$

Thus, the Cauchy-Riemann equations are satisfied, and $f$ is holomorphic.

If instead $f_2$ is a rotation by $-\pi/2$ of $f_1$, a similar argument shows $f$ is antiholomorphic. Therefore, for every point $z \in D$, either $f$ is holomorphic or $f$ is antiholomorphic.

To conclude that only one of these cases can be true on all of $D$, we need to apply the identity theorem (which we will see later). For now, let us assume the following fact: All zeros of a non-constant holomorphic function on a connected region are isolated.

Let $J(z) = \det(Df)$ denote the determinant of the Jacobian of $f$. Since $f \in C^2$, we have that $J \in C^1$. If $z_0 \in D$ is such that $f$ is holomorphic, then by the C-R equations,

$$J(z_0) = u_x v_y - u_y v_x = u_x(u_x) - u_y(-u_y) = u_x^2 + u_y^2 = \lambda(z_0) \geq 0.$$

Likewise, if $z_1 \in D$ is such that $f$ is antiholomorphic,

$$J(z_1) = -(u_x^2 + u_y^2) = -\lambda(z_1) \leq 0.$$

Since $D$ is connected, we can apply the intermediate value theorem to conclude that $J$ attains zero at least once. However, if $\lambda(z^*) = 0$, this implies that

$$u_x,\ u_y,\ v_x,\ v_y = 0.$$

Let us deal with this more precisely. Let

$$\begin{aligned}U = \{z \in D : J(z) > 0\}\\ V = \{z \in D : J(z) < 0\}.\end{aligned}$$

Note that, by the intermediate value theorem, $U$ and $V$ cannot consist of isolated points. Moreover, by continuity, $U$ and $V$ are open. Define also

$$\Gamma = \{z \in D : J(z) = 0\}$$

If $\Gamma$ contained an open set, then by analytic continuation $f$ would be constant on $D$, so we can assume this is not the case. Assume $U$ and $V$ are both non-empty. Since $D$ is connected, one cannot pass from $U$ to $V$ without intersecting $\Gamma$.

Let $g(z) = f'(z)$ for all $z \in D$. Then $g$ is holomorphic in $U$, continuous on $\overline{U}$, and vanishes on $\Gamma \cap \partial U$. If $\Gamma$ has an accumulation point, then the identity theorem would ensure that $g$ vanishes on all of $U$, so that $f$ is constant. Thus, $\Gamma$ can only consist of isolated points. But then $D \backslash \Gamma$ would still be connected, contradicting the intermediate value theorem applied to $J$ (since then the image would be the union of the disjoint intervals $(-\infty, 0)$ and $(0, \infty)$!). Hence, either $U = D$ or $V = D$.

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**Example 1.8 (September 2024, Problem 5):** Suppose $f \colon D \to \mathbb{C}$ is harmonic, $0 \notin f(D)$, and $1/f$ is also harmonic. Show that $f$ or $\overline{f}$ is holomorphic.

*Solution:* Recall that $f$ is harmonic if

$$\Delta f = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} f = 0,$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

Computing $\Delta(1/f)$, we find that

$$0 = \Delta\left(\frac{1}{f}\right) = \frac{2}{f^3}\left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}}\right) - \frac{1}{f^2}\Delta f = \frac{2}{f^3}\left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}}\right).$$

Thus, given $z \in D$, either $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \overline{z}} = 0$. By the identity theorem, exactly one of these must hold for all $z \in D$. That is, by the C-R equations, either $f$ is antiholomorphic (the first case) or holomorphic (the second case).

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**Exercise 1.3 (September 2023, Problem 4):** A complex function $w = W(z)$ can be expressed in polar coordinates: if $w = se^{i\gamma}$ and $z = re^{i\theta}$, then the two real functions of two real variables

$$s = S(r, \theta), \quad \gamma = \Gamma(r, \theta)$$

specify $w = W(z)$. Without changing to Cartesian coordinates, find necessary conditions on $S$ and $\Gamma$ for $W$ to have a derivative $W'$ at any point $z \neq 0$.

## 1.3 Elementary Functions

Let us make a quick detour to discuss how the exponential, logarithmic, and trigonometric functions generalize to the complex plane.

### 1.3.1 The complex logarithm

We already defined the complex exponential $e^z = e^x e^{iy}$. However, $e^z$ is periodic with period $2\pi i$, so it is not entirely clear how to define the logarithm such that the logarithm is not multivalued. We define, for $z \neq 0$,

$$\log(z) = \log(|z|) + i\arg(z) = \log(|z|) + i(\mathrm{Arg}(z) + 2\pi k) \quad \forall k \in \mathbb{Z}.$$

Then it follows that $e^{\log(z)} = z$. However,

$$\log(e^z) = z + 2\pi k i \quad \forall k \in \mathbb{Z}.$$

Therefore, we define the *principal value of the logarithm* as

$$\mathrm{Log}(z) = \log(|z|) + i\mathrm{Arg}(z).$$

The function $\mathrm{Log}(z)$ is then well-defined and single-valued for $z \neq 0$. It is straightforward to verify that $\mathrm{Log}(z)$ is continuous on the domain

$$\mathcal{D} = \{z = re^{i\theta} \in \mathbb{C} : r > 0,\ -\pi < \theta < \pi\}.$$

Moreover, one can check that $\mathrm{Log}(z)$ satisfies the C-R equations on $\mathcal{D}$, so it is also holomorphic. The derivative of $\mathrm{Log}(z)$ is, as one might expect, $1/z$. More generally, $\log(z)$ is analytic on the domain

$$\mathcal{D}_\alpha = \{z = re^{i\theta} \in \mathbb{C} : r > 0,\ \alpha < \theta < \alpha + 2\pi\}.$$

The line $\theta = \alpha$ is called a *branch cut* of the logarithm, and this definition extends naturally to other multi-valued functions. An single-valued analytic function $F$ that extends to a multi-valued function $f$ is called a *branch* of $f$. The principal value of the logarithm is often called the principal branch of $\log(z)$.

It is now reasonable to define exponentiation of a complex number by another complex number, via $z^c = e^{c\log(z)}$.

**Example 1.9:** Express $(1+i)^{-i}$ in the form $x + iy$ for some real $x, y$.

*Solution:* We write

$$(1+i)^{-i} = e^{-i\log(1+i)} = e^{-i(\log\left(\sqrt{2}\right)+i(\pi/4+2\pi k))} = e^{\pi/4+2\pi k}\left(\cos\left(\log\sqrt{2}\right) - i\sin\left(\log\sqrt{2}\right)\right).$$

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**Exercise 1.4 (September 2002, Problem 1):** Find the real and imaginary parts of the complex number $(1+i)^i$.

**Exercise 1.5 (September 2005, Problem 1):**
(a) Determine all complex numbers $z$ such that $i^z$ has a finite numbers of values.
(b) Same question for $i^{i^z}$.

### 1.3.2 Trigonometric & hyperbolic functions

We define the complex sine and cosine functions by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

and the hyperbolic sine and cosine by

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}, \quad \cosh(z) = \frac{e^{z} + e^{-z}}{2}.$$

A direct consequence of these definitions are the identities

$$\sin(z) = -i\sinh(iz), \;\; \cos(z) = \cosh(iz), \;\; \sinh(z) = -i\sin(iz), \;\; \cosh(z) = \cos(iz).$$

## 1.4 Power Series

Recall that an analytic function on $\mathbb{R}$ is any function that can locally be defined as a convergent power series, and any such function is infinitely differentiable. We can extend this definition to complex functions.

A *power series* is a series expansion of the form

$$\sum_{k=0}^{\infty} a_k z^k$$

for some coefficients $a_k \in \mathbb{C}$.

**Theorem 1.4 (Radius of Convergence)**

For any power series $\sum_{k=0}^{\infty} a_k z^k$, there exists a *radius of convergence* $R \in [0, \infty]$ such that the series converges in the open disk $|z| < R$ and diverges in the open region $|z| > R$. Moreover, $R$ can be found via the formulas:

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n} \qquad \text{(Root Test)}$$

$$1/R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{(Ratio Test)}$$

One can show that every power series defined a holomorphic function on its radius of convergence. More generally, any analytic function on an open set is holomorphic on that set. The derivative of a power series is, of course, obtained by term-by-term differentiation. However, the proof is not entirely trivial. Note that holomorphic functions are also analytic, but the proof relies on integration results concerning holomorphic functions which we will cover in Day 2.

**Example 1.10 (January 2016, Problem 2):** Consider the power series

$$f(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k}.$$

Determine the radius of convergence of $f(z)$. Does $f$ have an analytic continuation to any region beyond this disk of convergence? Justify.

*Solution:* By the root test, we compute

$$1/R = \limsup_{n\to\infty} |a_k|^{1/k} = 1,$$

so $f(z)$ converges in the disk $|z| < 1$. Observe now that

$$f(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k} = \sum_{k=0}^{\infty} (-z^2)^k = \frac{1}{1+z^2}.$$

The function $g(z) = 1/(1+z^2)$, defined on $\mathbb{C}\backslash\{i, -i\}$ then agrees with $f(z)$ on the disk $|z| < 1$, so by analytic continuation (the identity theorem), $f$ can be extended to $g(z)$.

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**Exercise 1.6 (September 2003, Problem 2):** By analytic continuation beyond the disk $|z+1| < 1$, determine $\lim_{z\to 1} f(z)$, where $f(z)$ is the analytic function defined in the disk $|z+1| < 1$ by

$$f(z) = \sum_{n=0}^{\infty} (n+1)(n+2)(z+1)^n.$$

# Day 2: Complex Integration & Residues

## 2.1 Contour Integrals

Of particular importance in complex analysis are integrals over closed curves. Before reviewing the key theorems about the integral of a holomorphic function over a closed curve, let us review contour integration.

We define a *curve* $C$ in $\mathbb{C}$ as a continuous function $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(t) = x(t)+iy(t)$. The point $\gamma(0)$ is called the *initial point* and $\gamma(1)$ the *terminal point*. If the initial and terminal points are the same, the curve is called *closed*. Note that we could replace $[0,1]$ in the definition with any real closed interval $[a,b]$ with $a < b$. The curve $\gamma$ is called *differentiable* if the component functions $x$ and $y$ are continuously differentiable functions of $t$, with

$$\gamma'(t) = x'(t) + iy'(t),$$

and such that $\gamma'(t) \neq 0$ for all $t \in [0,1]$. A *contour* is any piecewise differentiable curve. We are mostly interested in *simple curves*, that is, curves that do not intersect themselves. We use the term *positively oriented* to describe a curve that is oriented in the counterclockwise direction.

Given a contour $C$ parametrized by $\gamma$ on $[a,b]$, we define

$$\int_C f = \int_C f(z)\ dz := \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

**Exercise 2.1:** Compute the integral $I = \int_C \overline{z}\ dz$ when $C$ is given by the curve $z = 2e^{i\theta}$, where $\theta \in [-\pi/2, \pi/2]$.

**Example 2.1:** Compute the integral $I = \int_C \sqrt{z}\ dz$ where $C$ is the quarter-circle arc from $z = 1$ to $z = i$ and the integrand is taken with the principal branch.

*Solution:* The contour $C$ here is parametrized by the curve $\gamma(t) = e^{it}$ for $t \in [0, \pi/2]$. We have

$$I = \int_0^{\pi/2} e^{it/2}(ie^{it})\ dt = i\int_0^{\pi/2} e^{i(3t/2)}\ dt = \frac{2}{3}\left(-\frac{1}{\sqrt{2}} - 1 + i\frac{1}{\sqrt{2}}\right).$$

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**Exercise 2.2:** Redo Example 2.1 where the branch is given by $\pi < \theta < 3\pi$. Why is your answer different?

## 2.2 Analyticity and Integration

In this section, we will demonstrate how analyticity is a much stronger property than mere differentiability. It is likely that a significant portion of the written exam will require the use of the main results of this section, either explicitly or implicitly.

### 2.2.1 Antiderivatives & Cauchy's Theorem

We begin with the antiderivative theorem:

**Theorem 2.1 (Antiderivative Theorem)**

Let $f$ be continuous on a domain $D$. Then the following are equivalent:

(i) $f(z)$ has an antiderivative $F(z)$ in $D$;

(ii) The contour integral of $f$ from $z_1$ to $z_2$ in $D$ is independent of path. More specifically,

$$\int_\Gamma f = F(z_2) - F(z_1)$$

for any contour $\Gamma$ from $z_1$ to $z_2$;

(iii) For any closed contour $C$ in $D$,

$$\int_C f = 0.$$

**Example 2.2:** Evaluate the integral $\int_\gamma ze^z \, dz$, where $\gamma$ is the path parametrized by $\gamma(t) = t + i\sin^2(\pi t)$ for $t \in [0, 1]$.

*Solution:* The integrand as antiderivative $(z-1)e^z$, so we simply evaluate

$$\int_\gamma ze^z \, dz = (z-1)e^z|_{z=0}^{z=1} = 1.$$

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**Exercise 2.3:** Let $g(z) = 1/z$, defined on $D = \mathbb{C}\backslash\{0\}$. Show that $g$ does not have an antiderivative in $D$.

Perhaps the most important theorem in complex analysis is Cauchy's theorem (also known as Goursat's theorem, or Cauchy-Goursat):

**Theorem 2.2 (Cauchy-Goursat)**

Let $D \subseteq \mathbb{C}$ be a domain whose boundary is given by a simple contour $C = \partial D$. Suppose $f \in \mathcal{H}(D) \cap C(\overline{D})$. Then

$$\int_{\partial D} f = 0.$$

In particular, for any simple contour $C \subseteq D$, the integral of $f$ over $C$ vanishes. Combined with the antiderivative theorem, this shows that every holomorphic function on $D$ admits an antiderivative on $D$.

An immediate corollary of Cauchy's theorem is the "holes theorem," which allows one to reduce the integral of $f$ around $C$ to the sum of integrals around smaller contours $C_k$ which contain the "holes" of the domain $D$. More precisely, if $C_k$ are positively-oriented simple closed contours in $D$ such that $f$ is analytic in $D \setminus \bigcup_{k=1}^{N} C_k$, then

$$\int_{\partial D} f = \sum_{k=1}^{N} \int_{C_k} f.$$

### 2.2.2 Cauchy's Integral Formula

The first consequence of Cauchy's theorem is the Cauchy Integral Formula (CIF).

**Theorem 2.3 (Cauchy Integral Formula)**

Let $D$ be a domain bounded by a simple closed contour $\partial D$. Suppose $f \in \mathcal{H}(D) \cap C(\overline{D})$. Then for every $z_0 \in D$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\, dz.$$

The proof uses that $\int_{\partial D} dz/(z - z_0) = 2\pi i$ and considers the difference quotient of $f$ at $z_0$. The following generalization is also useful:

**Theorem 2.4 (Cauchy Derivative Formula)**

Under the same hypotheses as above,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

A consequence of the CIF is that all derivatives of analytic functions are also analytic. We also have the following simple estimate:

**Theorem 2.5 (Cauchy's Inequality)**

Suppose $f$ is holomorphic on $B(z_0, R)$ and continuous on $\overline{B(z_0, R)}$. Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n},$$

where $M_R = \sup_{\partial B(z_0,R)} |f(z)|$.

**Exercise 2.4 (Liouville's Theorem):** Any entire and bounded function is constant.

**Example 2.3 (September 2024, Problem 4):** Classify all entire functions $f(z)$ that satisfy $|f(z)| \leq \sqrt{1 + |z|}$ for all $z \in \mathbb{C}$.

*Solution:* Let $z_0 \in \mathbb{C}$ be given and take any circle $C_R$ centered at $z_0$. Let $z \in C_R$. Then

$$|z| \leq |z_0| + R,$$

and

$$|f(z)| \leq \sqrt{1 + R + |z_0|} \leq 1 + \sqrt{R} + \sqrt{|z_0|}.$$

Thus, by Cauchy's inequality

$$|f'(z_0)| \leq \frac{1}{R}\left(1 + \sqrt{R} + \sqrt{|z_0|}\right) \to 0 \text{ as } R \to \infty.$$

Thus, $f \equiv a$ for some $a \in \mathbb{C}$. Moreover,

$$|a| \leq \inf_{z \in \mathbb{C}} \sqrt{1 + |z|} = 1,$$

so any such $f$ is a constant within $\overline{B(0,1)} \subseteq \mathbb{C}$.

■

### 2.2.3 Morera's Theorem and Maximum Modulus Principle

Here we present two important consequences of Cauchy's theorem that may show up on the written exam. The first is combines the antiderivative theorem with the fact that the derivatives of holomorphic functions are holomorphic:

**Theorem 2.6 (Morera)**

Let $D$ be a domain. Suppose $f \in C(\overline{D})$ satisfies $\int_C f = 0$ for every simple closed contour $C$ in $D$. Then $f$ is analytic in $D$.

**Example 2.4:** Let $D$ be a domain and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on $D$. Suppose $f_n \to f$ uniformly on every compact subset of $D$. Show that $f$ is holomorphic on $D$.

*Solution:* Let $\overline{B} \subseteq D$ be any closed ball in $D$, and $\Delta \subseteq \overline{B}$ any triangle. Since each $f_n$ is holomorphic in $D$, we have $\int_\Delta f_n = 0$. Moreover, the convergence $f_n \to f$ is uniform, so $f$ is continuous, and

$$\int_\Delta f = \lim_{n \to \infty} \int f_n = 0.$$

By Morera's theorem, $f$ is analytic in $\overline{B}$, hence in $D$.

■

**Exercise 2.5 (January 2013 Problem 1(b)):** For which values of $z \in \mathbb{C}$ does the series

$$f(z) = \sum_{n=1}^{\infty} \frac{\cos(nz)}{e^n}$$

converge. For which values of $z$ is $f(z)$ analytic? *Hint: Use the previous example.*

It is also often convenient to rewrite the CIF in *Gauss mean value theorem* form: let $z_0 \in \mathbb{C}$ and let $C_\rho$ be a small positively-oriented circle $|z - z_0| = \rho$ around $z_0$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.$$

That is, for holomorphic $f$, it follows that $f(z_0)$ is the arithmetic mean of its values on $C_\rho$. This can be used to show that if $|f(z)| \leq |f(z_0)|$ in a neighbourhood of $z_0$, then $f$ is constant on this neighbourhood with value $f(z_0)$. The maximum modulus principle now follows:

**Theorem 2.7 (Maximum Modulus Principle)**

If $f$ is a non-constant holomorphic function on a domain $D$, then $|f(z)|$ does not attain a maximum in $D$.

A simple corollary of this is that if $D$ is bounded and $f$ is continuous on $\overline{D}$, then the extrema of $|f(z)|$ must lie on $\partial D$.

**Example 2.5 (September 2022, Problem 4):** Let $f$ be holomorphic on some open set containing the closed unit disk $|z| \leq 1$. Assume that

$$|1 - f(z)| \leq \left|e^{z-1}\right|$$

on $|z| = 1$. Prove that $1/2 \leq |f(0)| \leq 3/2$.

*Solution:* We have, for $|z| = 1$,

$$\left|\frac{1 - f(z)}{e^z}\right| \leq \frac{1}{e}.$$

Since $f$ is holomorphic in the disk, the maximum modulus principle applies to the above. In particular, for $z = 0$ we have

$$|1 - f(0)| \leq \frac{1}{e} \leq \frac{1}{2},$$

from which the desired bounds are derived.

■

**Example 2.6 (September 2017, Problem 5):** Let $f_k$ for $k = 1, \ldots, n$ be holomorphic in a domain $D$. Can the function

$$f(z) = \sum_{k=1}^{n} |f_k(z)|$$

have a strict local maximum in $D$? What about a strict local minimum?

*Solution:* We claim that $f$ cannot have a strict local maximum in $D$. Assume instead that $z_0$ is a maximizer of $f$. Let $C_\rho$ be a small circle around $z_0$ with radius $\rho$ such that $f(z) \leq f(z_0)$ for all $z \in C_\rho$. Then

$$f_k(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f_k(z_0 + \rho e^{it}) dt$$

for each $k = 1, \ldots, n$. Thus,

$$f(z_0) = \sum_{k=1}^{n} |f_k(z_0)| \leq \sum_{k=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \left| f_k(z_0 + \rho e^{it}) \right| \, dt = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{k=1}^{n} \left| f_k(z_0 + \rho e^{it}) \right| \right) \, dt$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it}) \, dt \leq f(z) < f(z_0),$$

a contradiction. Therefore, $f$ cannot have a strict local minimum in $D$.

A strict local minimum is fine. For instance, let $n = 1$, $f_1(z) = z$ and $f(z) = |z|$.

■

**Exercise 2.6 (January 2022, Problem 4):** Let $D$ be the unit disk centered at the origin and $\overline{D}$ its closure. Assume $f \in \mathbb{H}(D) \cap C(\overline{D})$ is non-constant such that $|f(z)| = 1$ for all $z \in \partial D$ (i.e. $|z| = 1$).

(a) Show that $f$ has a zero in $D$.

(b) Show that $f(D) = D$.

### 2.2.4 Analytic Continuation

Let us make precise the notion of analytic continuation, which we have used to solve some of the previous examples. The first result we require is the identity theorem.

**Theorem 2.8 (Identity Theorem)**

Let $h$ be holomorphic on a domain $D$. Then the following are equivalent:

(i) $h \equiv 0$ in $D$;

(ii) The set $D_0 = \{z \in \mathbb{C} : h(z) = 0\}$ has a limit point.

*Proof.* Choose $w_k \in D_0$ such that $w_k \to z_0$. We first show that $f$ vanishes in a disk $B \subseteq D$ around $z_0$. Consider the power series

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k.$$

If $f$ were not identically zero in $B$, then we can choose the smallest integer $N$ such that $a_N \neq 0$. Then

$$f(z) = a_N (z - z_0)^N (1 + g(z - z_0))$$

for some analytic function $g$ that vanishes as $z \to z_0$. Plugging the $w_k$ into this expression then provides a contradiction.

To conclude that $f$ vanishes on all of $D$, let $U$ denote the interior of the points where $f$ vanishes. We just showed that $U$ is non-empty, and, by definition, $U$ is open. But $U$ is also closed, since $f$ is continuous. Now, the domain $D$ is connected, so in fact $U = D$.

□

Now, if $f$ and $g$ are two analytic functions, the identity theorem can be readily applied to $h = f - g$ to show that $f = g$ on a domain $D$. If $f$ is defined on an open set $U \subseteq \mathbb{C}$, and $V \subseteq \mathbb{C}$ is another open set such that $U \subseteq V$. Then if $F$ is an analytic function on $V$ such that $F|_U = f$, we call $F$ an *analytic continuation* of $f$.

## 2.3 More Integration Exercises

**Exercise 2.7 (January 2025, Problem 1):** Let $C_2 = \{z \in \mathbb{C} : |z| = 2\}$. Use **CIF** to compute

$$\int_{C_2} \frac{1}{z^4 + 1} dz.$$

**Exercise 2.8 (September 2023, Problem 1):** State whether the following statements are true or false (a one or two-line justification for each will suffice).

(a) There exists a function which is analytic on the unit disk and such that

$$\int_{|z|=1/2} f = -1.$$

(b) There exists an unbounded entire function $f$ such that $\mathrm{Re}(f) \equiv -1$.

(c) The power series $\sum_{n \geq 0} n^n z^n$ converges at some complex number $z \neq 0$.

(d) There exists a nonzero function $f$ which is analytic on the unit disk, and such that $f^{(n)}(0) = 0$ for all $n \geq 0$.

(e) Assume $f$ is an entire function such that $|f(0)| \geq |f(z)|$ for all $z \in \mathbb{C}$, and let $f(z) = \sum_{n \geq 0} a_n z^n$ be its Taylor series. Then $a_5 = 0$.

**Exercise 2.9 (January 2011, Problem 3):** The function $f$ is analytic in the whole plane and have positive imaginary part. What can it be? What if all you know is that the imaginary part of $f$ tends to 0 at $\infty$?

## 2.4 Laurent Series

Any function $f$ which is holomorphic at a point $z_0$ admits a Taylor series at $z_0$. This fact is yet another consequence of the CIF. Thus, holomorphic and analytic are synonymous. A *Laurent series* generalizes the Taylor series to functions that are analytic except for at a point.

**Theorem 2.9**

Suppose $f$ is analytic throughout some annulus $R_1 < |z - z_0| < R_2$ centered at $z_0$, and let $C$ be any simple closed contour around $z_0$ contained in this domain. Then $f$ admits the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

in $R_1 < |z - z_0| < R_2$. The coefficients are given explicitly by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{1-n}}\, dz.$$

The proof, like for Taylor's theorem, relies on some CIF shenanigans. Computing the Laurent series in a given annulus is a basic exercise that often shows up on exams.

**Example 2.7 (September 2025, Problem 3):** Consider the complex function

$$f(z) = \frac{1}{z(z-1)(z-2)}.$$

Obtain the Laurent series expansion in the annulus $1 < |z| < 2$.

*Solution:* First, let us split up $f$ via partial fractions to obtain

$$f(z) = \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{2(2-z)}.$$

The first term is already of the form $1/z$, so we can leave it alone for now. Since $|z| > 1$, we rewrite the second term as

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1 - (1/z)} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=1}^{\infty} \frac{1}{z^n}.$$

The last term becomes

$$\frac{1}{2(2-z)} = \frac{1}{4(1-(z/2)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}}.$$

Overall, the Laurent series is

$$f(z) = \left(-\frac{1}{2}\right)\frac{1}{z} - \sum_{n=2}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+2}}.$$

■

**Exercise 2.10:** Find the Laurent series expansion of $f(z)$ from Example 2.7 in the regions $0 < |z| < 1$ and $|z| > 2$.

## 2.5 Residue Calculus

### 2.5.1 Cauchy's Residue Theorem

If $f$ is analytic on a domain $D$ except at finitely many points $z_1, \ldots, z_n$, we can define the *residue* of $f$ at the point $z_k$ by

$$\operatorname{Res}_{z=z_k} f(z) \coloneqq 2\pi i b_1^{(k)},$$

where $b_1^{(k)} = \int_{C_k} f$ is the first coefficient of the Laurent series expansion of $f$ at the point $z_k$. Here, $C_k$ is a positively oriented simple closed contour contained in $D$ that contains $z_k$. Thus,

$$\int_{C_k} f = 2\pi i \operatorname{Res}_{z=z_k} f(z).$$

**Theorem 2.10 (Cauchy's Residue Theorem)**

Let $D$ be a domain bounded by a (positively oriented) simple closed curve $\partial D$. Suppose $f$ is analytic in $D$ except at finitely many points $z_1, \ldots, z_n \in D$, and that $f$ is continuous on $\overline{D}$. Then

$$\int_{\partial D} f = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

This is, in effect, exactly the same thing as the previously discussed "holes theorem" (see Theorem 2.2).

**Exercise 2.11 (September 2023, Problem 3):** Consider the function

$$f(z) = \frac{\sin(z)}{z^3(z-1)^2}.$$

(a) Find the first two non-zero terms of the Laurent series of $f$ in the annulus

$0 < |z| < 1/2$, and compute $\int_{C_0} f$ where $C_0$ is the circle of radius $1/4$ centered at $z = 0$, oriented CCW.

(b) Repeat part (a) for the annulus $0 < |z - 1| < 1/2$ and $C_1$ the circle of radius $1/4$ centered at $z = 1$, oriented CCW.

### 2.5.2 Poles and zeros of analytic functions

If $f$ is analytic except at a point $z_0$, the part of the Laurent series of $f$ with negative powers of $(z - z_0)$,

$$\frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_k}{(z - z_0)^k} + \ldots$$

is called the *principal part*. If the principal part terminates at the power $k = N$, then the point $z_0$ is called a *pole of order $N$ of $f$*. In the case $N = 1$, $z_0$ is often called a *simple pole*. If each $b_k = 0$, then $z_0$ is called a *removable singularity*, and if the $b_k$ never terminate, $z_0$ is an *essential singularity*. Essential singularities are particularly strange (we will state a few theorems concerning them later). A particularly important example of a function with an essential singularity is $e^{1/z}$. An important piece of terminology is the term *meromorphic*, which describes a function that is holomorphic on its domain except at finitely many isolated singularities, all of which being poles.

A convenient characterization of poles that is often used when computing residues is the following:

**Theorem 2.11**

Let $f$ be analytic except at an isolated singular point $z_0$. Then $z_0$ is a pole of order $N$ if and only if

$$f(z) = \frac{\phi(z)}{(z - z_0)^N}$$

for some analytic function $\phi$ with $\phi(z_0) \neq 0$. Moreover, if $N = 1$,

$$\text{Res}_{z=z_0} f(z) = \phi(z_0),$$

and if $N > 1$,

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(N-1)}(z_0)}{(N-1)!}.$$

It is straightforward to define $\phi$ given the Laurent series of $f$ by setting

$$\phi(z) = \begin{cases} (z - z_0)^N f(z) & z \neq z_0 \\ b_N & z = z_0 \end{cases}.$$

**Exercise 2.12 (January 2025, Problem 1 (again)):** Let $C_2 = \{z \in \mathbb{C} : |z| = 2\}$. Use **residues** to compute

$$\int_{C_2} \frac{1}{z^4+1} dz.$$

Let us also quickly discuss the zeros of analytic functions so that we can state another nice formula for computing contour integrals. If $f(z_0) = 0$ and $N$ is the smallest integer such that $f^{(N)}(z_0) \neq 0$, then $z_0$ is called a *zero of order $N$ of $f$*.

**Theorem 2.12**

Let $f$ is analytic at $z_0$. Then $z_0$ is a zero of order $N$ of $f$ if and only if there is an analytic function $g$ with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^N g(z)$.

A nice corollary of this is that the zeros of analytic functions are isolated (Exercise: prove this). From this, it is easy to see that if $p, q$ are analytic functions such that $p(z_0) \neq 0$ and $q$ has a zero of order $N$ at $z_0$, then $p/q$ has a pole of order $N$ at $z_0$. Now, for simple poles, this provides a simple formula for finding residues:

**Theorem 2.13**

Let $p, q$ be analytic at $z_0$ such that $p(z_0) \neq 0$ and $q$ has a zero of order 1 at $z_0$. Then

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

**Example 2.8 (January 2012, Problem 2):** Compute

$$\int_C \frac{z\sec(z)}{(1-e^z)^2}\, dz,$$

where $C$ is the circle of radius 2 centered at the origin.

*Solution:* We first rewrite the integrand as

$$f(z) = \frac{z\sec(z)}{(1-e^z)^2} = \frac{z}{\cos(z)(1-e^z)^2},$$

from which we can identify $z = 0, \pm\pi/2$ as simple poles. We compute the residues at each pole:

For $\pi/2$, we compute

$$\text{Res}_{z=\pi/2} f(z) = \frac{\pi/2}{(-\sin(\pi/2)(1-e^{\pi/2})^2 - 2\cos(\pi/2)e^{\pi/2}(1-e^{\pi/2}))} = -\frac{\pi}{2(1-e^{\pi/2})^2}.$$

Likewise,

$$\text{Res}_{z=-\pi/2} f(z) = -\frac{\pi}{2(1-e^{-\pi/2})^2}.$$

For $z = 0$, we first observe that

$$\left(\frac{(1-e^z)^2}{z}\right)' = \frac{-2ze^z(1-e^z)-(1-e^z)^2}{z^2}.$$

Thus, after two applications of L'Hopital, we find that

$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{z^2}{-2ze^z(1-e^z)-(1-e^z)^2} = 1.$$

Overall,

$$\int_C \frac{z\sec(z)}{(1-e^z)^2}\, dz = 2\pi i\left(1 - \frac{\pi}{2(1-e^{\pi/2})^2} - \frac{\pi}{2(1-e^{-\pi/2})^2}\right).$$

■

# Day 3: Applications of Residues & Conformal Mappings

## 3.1 Computing Real Integrals via Residues

Computing a real integral via complex-analytic methods is possibly the most likely question to appear on the written exam. We will cover three examples, and a few more important exercises are left below those.

### 3.1.1 Three important examples

**Example 3.1 (September 2024, Problem 1):** Compute the following integral using the residue theorem:

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)^2}\, dx.$$

*Solution:* We consider the complexified (is this a word people use?) version of the integrand

$$f(z) = \frac{z}{(z^2+2z+2)^2},$$

and observe that $f$ has two poles, each of order 2, at

$$z_0 = -1+i, \quad z_1 = -1-i.$$

Let $C_R$ be the positively oriented semicircle of radius $R$ in the closed upper half-plane, $R \gg 1$. Moreover, let $\Gamma_R$ be the interval $[-R, R]$ on the real axis, and define $C = \Gamma_R \cup C_R$, oriented positively. We compute the residue of $f$ as $z_0$ as follows:

$$\begin{aligned}\operatorname{Res}_{z=z_0} f(z) = \int_C \frac{z/(z+1+i)^2}{(z-(-1+i))^2}\, dz &= 2\pi i \left(\frac{z}{(z+1+i)^2}\right)'\Big|_{z=-1+i} \\ &= 2\pi i \frac{(2i)^2 - 2(-1+i)(2i)}{(2i)^4} \\ &= -\frac{\pi}{2},\end{aligned}$$

by CIF. The next step is to show that the part on the arc of the circle vanishes as we take $R$ to infinity. To this end, fix $z \in \mathbb{C}$ with $|z| = R$. Then

$$|f(z)| \le \frac{R}{(R^2-2R-2)^2},$$

where we have applied the triangle inequality. Thus,

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\, dz \right| \le \lim_{R\to\infty} \pi \frac{R^2}{(R^2-2R-2)^2} = 0.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)^2}\, dx = \int_C \frac{z}{(z^2+2z+2)^2}\, dz = -\frac{\pi}{2}.$$

■

> **Example 3.2 (January 2007, Problem 1):** Show that $\sum_{n\geq 1} 1/n^2 = \pi^2/6$ by integrating $1/(z^2(e^{2\pi iz}-1)$ along the boundary of a suitable box.

*Solution:* First note that
$$f(z) = \frac{1}{z^2(e^{2\pi iz}-1)}$$
has a triple pole at $z$ and simple poles at $z = k$ for every integer $k \neq 0$. It is a straightforward computation to show that
$$\operatorname{Res}_{z=k} f(z) = \frac{1}{2\pi ik^2}.$$
We must choose a contour that avoids all the roots. To this end, consider the box

(I)
$$z = x - i\left(N + \frac{1}{2}\right),$$

(II)
$$z = \left(N + \frac{1}{2}\right) + iy,$$

(III)
$$z = x + i\left(N + \frac{1}{2}\right),$$

(IV)
$$z = -\left(N + \frac{1}{2}\right) + iy,$$

where
$$x, y \in \left[-\left(N + \frac{1}{2}\right), N + \frac{1}{2}\right]$$
and the box $C$ is to be oriented positively.

On (I) and (III), observe that
$$\lim_{N\to\infty}\left|\int_\Gamma f(z)\, dz\right| \leq \lim_{N\to\infty} 2\left(N + \frac{1}{2}\right)\frac{1}{(N+1/2)^2(e^{-2\pi(N+1/2)}-1)} \to 0,$$
by the triangle inequality. On the contours (II) and (IV), we find
$$e^{2\pi iz} = e^{2\pi iN}e^{\pi i}e^{-2\pi y} = -e^{-2\pi y}.$$

Then the contour integrals over (II) and (IV) again decay like $1/N$. By Cauchy's theorem
$$2\sum_{n=1}^{\infty}\frac{1}{n^2} + 2\pi i \operatorname{Res}_{z=0} f(z) = 0.$$

To compute $\operatorname{Res}_{z=0} f(z)$, one can do series division to find

$$\frac{1}{z^2(e^{2\pi iz}-1)} = \frac{1}{2\pi iz^3}\cdot\frac{1}{1+\pi iz+\frac{(2\pi i)^2}{6}z^2+\dots} = \frac{1}{(2\pi i)z^3} - \frac{1}{2z^2} + \frac{i\pi}{6}\cdot\frac{1}{z} + \dots,$$

from which we identify $\operatorname{Res}_{z=0} f(z) = (\pi i)/6$. ■

**Example 3.3 (September 2012, Problem 3):** Show that

$$I := \int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}$$

by integrating around the contour which consists of arc of the circle of radius $R$ for $0 \le \theta \le 2\pi/3$, where it becomes the line segment $re^{i2\pi/3}$ for $0 \le r \le R$.

*Solution:* Let $f(z) = 1/(1+z^3)$, which has poles at $z = -1, e^{i\pi/3}, e^{-i\pi/3}$. As suggested, we split the contour into the arc or the circle of radius $R$, denoted $C_R$, for $0 \le \theta \le 2\pi/3$, the ray $\Gamma_R$ parametrized by $re^{2\pi i/3}$ for $0 \le r \le R$, and the interval $I_R = [0, R]$ on the real line. Let $C = C_R \cup (-\Gamma_R) \cup I_R$, oriented positively. Then

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz - \int_{\Gamma_R} f(z)\,dz + \int_{I_R} f(z)\,dz.$$

First, observe that the only pole of $f$ interior to $C$ is $z = e^{i\pi/3}$. Computing the residue, we find that

$$2\pi i \operatorname{Res}_{z=e^{i\pi/3}} f(z) = \frac{2\pi i}{3(e^{i\pi/3})^2} = \frac{2\pi i}{3e^{2\pi i/3}}.$$

We now evaluate $\int_{\Gamma_R} f$ using the parametrization $re^{2\pi i/3}$ as follows:

$$\int_{\Gamma_R} \frac{1}{1+z^3}\,dz = e^{2\pi i/3}\int_0^R \frac{1}{1+r^3}\,dr \to e^{2\pi i/3} I$$

in the limit $R \to \infty$. Lastly, observe that

$$\left|\int_{C_R} f(z)\,dz\right| \le C\frac{R}{R^3-1} \to 0$$

as $R \to \infty$. Thus,

$$\frac{2\pi i}{3e^{2\pi i/3}} = 2\pi i \operatorname{Res}_{z=e^{i\pi/3}} f(z) = \int_C f(z)\,dz = (1-e^{2\pi i/3})I.$$

Simplifying, this becomes $I = 2\pi/(3\sqrt{3})$.

■

### 3.1.2 Exercises on real integrals

**Exercise 3.1 (September 2016, Problem 3):** For any real number $p > 1$, calculate

$$\int_0^\infty \frac{1}{x^p + 1}\, dx.$$

**Exercise 3.2 (September 2007, Problem 2):** Calculate the integral

$$I = \int_0^\infty \frac{\cos(x)}{x^2 + 1}\, dx.$$

**Exercise 3.3 (January 2011, Problem 1):** Calculate the integral

$$I = \int_{-\infty}^\infty \frac{1}{x^4 + 3x^2 + 4}\, dx.$$

**Exercise 3.4 (January 2017, Problem 5(b)):** Let $n$ be an arbitrary positive integer. Evaluate

$$\int_0^\infty \frac{1}{x^{2n} + x^{-2n}}\, dx.$$

*Hint: Use a suitable shaped and positioned "pizza slice" contour.*

## 3.2 Argument Principle & Rouché's Theorem

One of the most frequent questions to appear on the written exam is to determine the number of roots of a function inside some disk. The solution to this almost always involves applying Rouché's theorem, which is a consequence of the argument principle.

**Theorem 3.1 (Argument Principle)**

Suppose $f$ is meromorphic on a domain $D$ and let $C$ be a circle such that $C$ and its interior are inside $D$, and such that $f$ has no poles or zeros on $C$. Then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = Z - P,$$

where $Z, P$ are the number of zeros and poles of $f$, respectively, counting multiplicity.

The idea behind this result is that the integrand is the logarithmic derivative, so the integral of this quantity measures the change in the argument of $f$ as $z$ traverses the curve $C$. If $f$ is meromorphic, then $f'/f$ will have simple poles at the zeros and poles of $f$, and so the residue theorem may be applied.

**Theorem 3.2 (Rouché)**

Let $f, g$ be holomorphic in the domain $D$, which contains a circle $C$ and the interior of $C$. Suppose that

$$|f(z)| > |g(z)| \quad \forall z \in C.$$

Then $f$ and $f + g$ have the same number of zeros inside $C$.

The proof follows from the argument principle applied to the family of functions $f_t = f + tg$ for $t \in [0, 1]$. The idea is to show that the function

$$Z_t = \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)}\, dz$$

is constant in $t$. The details are left as an exercise.

**Example 3.4 (September 2024, Problem 3):** Determine the number of zeros of $z^8 - 2024z^4 + z + 1$ on the unit disk in $\mathbb{C}$.

*Solution:* We apply Rouché's theorem to the functions $f(z) = -2024z^4$ and $g(z) = z^8 + z + 1$ to find that $f + g$ has four zeros in the unit disk (counting multiplicities). ■

**Example 3.5 (January 2005, Problem 2):** Consider the polynomial of degree $n$,

$$P_n(z) = z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}.$$

Arrange its $n$ roots $(z_{n,1}, \ldots, z_{n,n})$ in increasing order of magnitude

$$0 = |z_{n,1}| \leq |z_{n,2}| \leq \ldots \leq |z_{n,n}|.$$

What happens to $z_{n,2}$ as $n \to \infty$? Why?

*Solution:* Note first that $P_n(z)$ is the partial sum of $e^z - 1$, which has roots at integer multiples of $2\pi i$. Thus, we would expect $|z_{n,2}| \to 2\pi$ as $n \to \infty$.

To see this, let us first apply Rouché to upper bound $|z_{n,2}|$. Let $m = \inf_{|z|=3\pi} |f(z)|$, and note that $m > 0$, as $f$ is non-zero on the circle $|z| = 3\pi$. Now write

$$P_n(z) = f(z) + (P_n(z) - f(z)).$$

By convergence of the partial sums on $\mathbb{C}$, for sufficiently large $n$, it follows that

$$|P_n(z) - f(z)| \leq \frac{m}{2}$$

for all $z$ interior and on the circle $|z| = 3\pi$. Thus, Rouché gives that $P_n(z)$ and $f(z)$ share the same number of zeros in $|z| = 3\pi$. However, we know the zeros of $f$ exactly! The only zeros of $f$ inside this circle are $z = 0, \pm 2\pi i$. Thus, for sufficiently large $n$ it follows that $P_n(z)$ has three zeros in $|z| < 3\pi$. One of these zeros is $z = 0$.

Now consider the region $|z - 2\pi i| \leq \delta$ for $\delta > 0$ small and arbitrary. Letting $m_\delta$ denote the infimum of $f$ over this circle, we can apply the same argument above to show that, for $\delta$ sufficiently small, $P_n$ has one zero in this disk. As $\delta$ shrinks, this zero approaches $2\pi i$. The same argument holds for $|z + 2\pi i| \leq \delta$, so that indeed $|z_{n,2}| \to 2\pi$ as $n \to \infty$.

■

**Exercise 3.5 (January 2023, Problem 2):** Let $p(z) = z^3 - 8z^2 + 7z + 2$. How many roots of $p$ (counting multiplicities) lie in the disk $|z| < 3$?

**Exercise 3.6 (September 2014, Problem 4):** How many roots does the equation $z^4 - 6z + 3 = 0$ have in the annulus $1 < |z| < 2$?

## 3.3 More on Singularities

### 3.3.1 Essential Singularities

There are some facts about essential singularities you should know for the written exam (sometimes these make a proof more-or-less immediate!) The first result is the Casorati-Weierstrass theorem. The key ingredient in the proof is a result from Riemann:

**Theorem 3.3 (Riemann's theorem on removable singularities)**

If $f$ is holomorphic and bounded on a punctured disk $D^*$ around a point $z_0$, then $z_0$ is a removable singularity of $f$.

A simple consequence of this is that, given an isolated singularity $z_0$ of $f$, then $z_0$ is a pole of $f$ if and only if $|f(z)| \to \infty$ as $z \to z_0$.

**Theorem 3.4 (Casorati-Weierstrass)**

Let $f$ be holomorphic in a punctured disk $D^*$ centered around $z_0$ and suppose $f$ has an essential singularity at $z_0$. Then $f(D^*)$ is dense in $\mathbb{C}$.

*Proof.* Assume instead $f(D^*)$ is not dense in $\mathbb{C}$. Then there exist $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta$$

for all $z \in D^*$. Now, define the auxiliary function

$$g(z) = \frac{1}{f(z) - w}.$$

Then $g$ is holomorphic and bounded in $D^*$, hence $z_0$ is a removable singularity of $g$. If $g(z_0) \neq 0$, then $f(z) - w$ would be holomorphic at $z_0$, which is a contradiction. Thus, $g(z_0) = 0$, so that $f(z) - w$ has a pole at $z_0$, a contradiction.

□

The Great Picard Theorem is a significant strengthening of the above result:

**Theorem 3.5 (Great Picard)**

Let $f$ be holomorphic on a punctured disk $D^*$ centered at $z_0$ and suppose $f$ has an essential singularity at $z_0$. Then $f(D^*)$ is either all of $\mathbb{C}$ or $\mathbb{C}\backslash\{w\}$ for some $w \in \mathbb{C}$, where every value of $f(D^*)$ is taken on infinitely often.

The weaker version of this theorem is an important result concerning entire functions:

**Theorem 3.6 (Little Picard)**

Suppose $f$ is non-constant and entire. Then either $f$ attains every value of $\mathbb{C}$ or $\mathbb{C}\backslash\{w\}$ for some $w \in \mathbb{C}$.

Before seeing an example, let's quickly review how singularities at infinity work.

### 3.3.2 Singularities at Infinity

We can also define the notion of *singularities at infinity* of a function $f$ by looking at the type of singularity of $g(z) = f(1/z)$ at $z = 0$. A *removable singularity at infinity* would then be characterized by a function $f$ such that $\lim_{z\to\infty} f(z) = L \in \mathbb{C}$. In particular, the Laurent series of $f$ at $z = 0$ has no positive powers of $z$. Likewise, a *pole of $f$ at infinity* occurs when $\lim_{z\to\infty} f(z) = \infty$, and the order of the pole is given by the largest positive power in the Laurent series expansion of $f$. An *essential singularity at infinity* then means that the limit $\lim_{z\to\infty} f(z)$ does not exist, corresponding to infinitely many terms with positive power in the Laurent series of $f$.

We can now classify entire functions by their behaviour at infinity. By Liouville's theorem, an entire function with a removable singularity at infinity is constant. If an entire function has a pole of order $N$ at infinity, then that function is a polynomial of degree $N$. Otherwise, entire functions that have an essential singularity at infinity are called *transcendental*.

**Example 3.6 (September 2025, Problem 1):** Let $f$ be non-constant and entire. Prove that for every complex number $c$ there is an infinite sequence $\{z_n\}$ such that $\lim_{n\to\infty} f(z_n) = c$.

*Solution 1 (Little Picard):* This is essentially an immediate consequence of Little Picard, since the closure of $\mathbb{C}$ minus a point is just $\mathbb{C}$.

*Solution 2 (Casorati-Weierstrass):* Since $f$ is non-constant and entire, either $f$ is a polynomial or $f$ is transcendental. If the former is true, then $f$ is surjective (this is a consequence of the Fundamental Theorem of Algebra), and we are done. Otherwise, $f$ is transcendental, so we can apply Casorati-Weierstrass to conclude, where the theorem is applied to a neighbourhood of infinity.

There is a third solution via Liouville but we will see this solution later.

■

**Example 3.7 (January 2007, Problem 4):** What is the most general entire function that takes each value of $\mathbb{C}$ once and only once?

*Solution:* Let $f$ be given as in the question statement. Since $f$ is entire, the principal part of its Laurent series is trivial. It suffices to study the type of singularity of $f$ at infinity. By Big Picard, $f$ cannot have an essential singularity at infinity, or else it takes on each value of $\mathbb{C}$ (except maybe one) infinitely often. Thus, $f(1/z)$ cannot have infinitely terms with negative power in its Laurent series. Therefore, $f$ is a polynomial. Since $f$ attains 0 only once, it only has one root. Thus, we can write $f(z) = a(z - z_0)^m$ for some $a \in \mathbb{C}$ and $m \in \mathbb{N}$. If $m > 1$, then the equation $f(z) = w$ has more than one solution for each $w \in \mathbb{C}$. Therefore, $f$ is any linear polynomial.

■

**Exercise 3.7 (September 2012, Problem 5):** Let $f$ be an entire function which does not take any value more than 3 times. What can it be?

## 3.4 Definitions of Conformal Mappings

Let us first introduce some terminology. A mapping $f : U \to V$, $U, V \subseteq \mathbb{C}$ open, is called a *conformal map* (or a *biholomorphism*) if $f$ is bijective and holomorphic. If a conformal map $U \to V$ exists, we call $U$ and $V$ *conformally equivalent*.

The first fundamental result concerning conformal maps is the following:

**Theorem 3.7**

Let $f$ be a holomorphic and injective map from an open set $U$ to the open set $V$. Then $f'(z) \neq 0$ for all $z \in U$.

In particular, the inverse of $f$, defined on $f(U) \subseteq V$, is holomorphic, via the formula

$$(f^{-1})'(w) = 1/f'(f^{-1}(w)).$$

Therefore, any conformal map has a holomorphic inverse.

Sometimes, the definition of a holomorphic map $f : U \to V$ being conformal is simply the condition that $f'(z) \neq 0$ for all $z \in U$. Any such map will preserve angles, and moreover is locally bijective. However, we will follow the convention in [2] for these notes and assume that bijectiveness is cooked into the definition of conformal. If you wish, you can solve the following exercise:

**Exercise 3.8 (Stein & Shakarchi [2], Chapter 8, Exercise 1):** A holomorphic map $f\colon U \to V$ is a *local bijection* on $U$ if for every $z \in U$ there is a disk $D \subseteq U$ centered at $z$ so that $f|_D\colon D \to f(D)$ is a bijection.
Prove that $f\colon U \to V$ is a local bijection if and only if $f'(z) \neq 0$ for all $z \in U$. *Hint: Use Rouché's theorem.*

**Example 3.8 (January 2016, Problem 5):** Consider the map $w = f(z)$ defined by

$$w = \frac{1}{2}\left(3z + \frac{1}{z}\right).$$

What is the image of the unit circle under this map? Where is this map conformal?

*Solution:* Given $z = e^{i\theta}$, we compute

$$f(z) = \frac{1}{2}\big(3e^{i\theta} + e^{-i\theta}\big) = 2\cos(\theta) + i\sin(\theta),$$

so the image of the unit circle is an ellipse intercepting the real axis at $z = \pm 2$ and imaginary axis at $z = \pm i$.

To determine where $f(z)$ is conformal, let us first find the points where $f'(z) \neq 0$. The function has a pole at $z = 0$, so $f$ cannot be conformal there. We compute

$$f'(z) = \frac{1}{2}\left(3 - \frac{1}{z^2}\right) = 0,$$

so that $z = \pm 1/\sqrt{3}$ and $z = 0$ are the points where $f$ is not locally conformal. Now, our definition of conformality insists that $f$ is a bijection. If we just define $f$ on the domain $\mathbb{C}\backslash\{0, \pm 1/\sqrt{3}\}$, then $f$ is actually a 2-to-1 mapping, since $z$ and $1/(3z)$ map to the same point. Thus, we must restrict $f$ to subdomains of one of the following two regions to ensure it is conformal:

$$\left\{z \in \mathbb{C} : |z| > \frac{1}{\sqrt{3}}\right\}, \quad \left\{z \in \mathbb{C} : 0 < |z| < \frac{1}{\sqrt{3}}\right\}.$$

■

The mapping considered in the previous example comes up quite frequently, so it is worth a brief comment to fully understand it. Namely, the mapping

$$z \longmapsto z + \frac{1}{z}$$

is called the *Joukowski transformation*.

**Example 3.9:** Show that the Joukowski transformation maps the upper half-disk to the lower half-plane.

*Solution:* This is easiest to see in polar coordinates, as done in the above example. Indeed,

$$re^{i\theta} + r^{-1}e^{-i\theta} = \left(r + \frac{1}{r}\right)\cos(\theta) + i\left(r - \frac{1}{r}\right)\sin(\theta).$$

Now, in the upper half-disk, we have $\theta \in (0, \pi)$, so that $\sin(\theta) > 0$. However, for $r < 1$, the coefficient $r - r^{-1}$ is negative. Thus, the upper half-disk is mapped into the lower half-plane.

As an **exercise**, show that the transformation is bijective and conclude it is conformal from the upper half-disk to the lower half-plane.

■

## 3.5 Linear Fractional Transformations

One important class of conformal mappings are the *linear fractional transformations* (also known as *Möbius transformations*), defined by

$$w = F(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. The inverse of $F$ is given by

$$z = G(w) = \frac{dw - b}{-cw + a},$$

and hence is also a linear fractional transformation.

**Example 3.10 (Mapping the upper half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$):** Define

$$F(z) = \frac{i - z}{i + z}.$$

Show that $F$ is a conformal map from $\mathbb{H}$ to $\mathbb{D}$, with inverse

$$G(w) = i\frac{1 - w}{1 + w}.$$

*Solution:* Any point in $\mathbb{H}$ is closer to $i$ than $-i$, so indeed $F$ is a well-defined mapping from $\mathbb{H}$ to $\mathbb{D}$. Now, if $w = u + iv \in \mathbb{D}$, then

$$\text{Im}(G(w)) = \text{Re}\left(\frac{1 - u - iv}{1 + u + iv}\right) = \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0,$$

since $|w| < 1$. To conclude, it is straightforward to check that $F(G(w)) = w$ for all $w \in \mathbb{D}$.

■

**Example 3.11 (January 2021, Problem 2):** Let $f(z) = \exp\left(\frac{1+z}{1-z}\right)$ for $z \neq 1$.
(a) Find all zeros of $f$.
(b) Where is $f$ holomorphic?
(c) Is $f$ bounded on the unit circle, from which $z = 1$ is removed?
(d) If $f$ bounded on the open unit disk?

*Solution:* Note that $(1+z)/(1-z)$ is a linear fractional transformation, mapping $\mathbb{D}$ to the right open half-plane $\text{Re}(z) > 0$. For (a), since $e^z$ has no roots, there are no zeros of $f$. Moreover, since $f$ is the composition of a conformal mapping and an entire function, $f$ is holomorphic everywhere except $z = 1$.

Under the transformation $(1+z)/(1-z)$, the unit circle without $z = 1$ is mapped to the imaginary axis, which in turn maps to the unit circle under the exponential map. Therefore, $f$ is bounded on the unit circle without $z = 1$.

However, $f$ is not bounded on the open unit disk. Indeed, the transformation $(1+z)/(1-z)$ maps the open unit disk to the right half-plane $\text{Re}(z) > 0$, which is then mapped to the exterior of $\overline{\mathbb{D}}$.

■

**Exercise 3.9 (September 2008, Problem 4):**
(a) Prove that, if $z$ and $w$ are complex numbers and $|w| = 1$, then

$$\left|\frac{z-w}{1-\overline{z}w}\right| = 1.$$

(b) Prove that, if $|z| < 1$, $|w| < 1$, then

$$\left|\frac{z-w}{1-\overline{z}w}\right| < 1.$$

**Exercise 3.10:** Let $\text{SL}(2,\mathbb{R})$ denote the space of $2 \times 2$ matrices with determinant equal to 1. To each

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{R})$$

is a corresponding linear fractional transformation $T_A(z) = (az+b)/(cz+d)$. Prove that the number and location of the fixed points of $T_A$ (i.e. $T_A(z) = z$) in the extended plane $\mathbb{C} \cup \{\infty\}$ are determined by the trace of $A$, $\tau = \text{Tr}(A)$ as follows:
(a) $|\tau| < 2$ if and only if $T_A$ has two fixed points, which are complex conjugates.
(b) $|\tau| = 2$ if and only if $T_A$ has one fixed point, which lies on $\mathbb{R} \cup \{\infty\}$.
(c) $|\tau| > 2$ if and only if $T_A$ has two distinct fixed points, which lie on $\mathbb{R} \cup \{\infty\}$.
*Hint: Look at the discriminant of a certain polynomial.*

# Day 4: Riemann Mapping Theorem & Practice Exams

## 4.1 The Schwarz Lemma

### 4.1.1 The lemma

The Schwarz lemma is one of the most important results concerning conformal mappings, and is a key ingredient in proving the Riemann mapping theorem.

**Theorem 4.1 (Schwarz Lemma)**

Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = 0$. Then

(a) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$;

(b) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$, then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$ (i.e. $f$ is a rotation);

(c) $|f'(0)| \le 1$, and $f$ is a rotation if equality holds.

**Example 4.1 (January 2025, Problem 3(b)):** Let $f\colon \mathbb{H} \to \mathbb{H}$ be holomorphic such that $f(i) = i$. Prove that $|f'(i)| \le 1$. Can you classify the maps for which equality holds?

*Solution:* Define $g\colon \mathbb{D} \to \mathbb{D}$ by

$$g(z) = (F \circ f \circ G)(z),$$

where $F$ and $G$ are as in Example 3.10. Then $g$ is holomorphic, and

$$g(0) = F(f(G(0))) = F(f(i)) = F(i) = 0.$$

By the Schwarz lemma, we have $|g'(0)| \le 1$. However, we compute

$$g'(z) = F'(f(G(z))) \cdot f'(G(z)) \cdot G'(z),$$

so that

$$g'(0) = F'(i)f'(i)(F^{-1})'(0) = F'(i)f'(i)\frac{1}{F'(F^{-1}(0))} = f'(i),$$

so that $|f'(i)| \le 1$. If equality were to hold, then the Schwarz lemma says that $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. In particular, $f(z) = G(e^{i\theta} F(z))$.

■

### 4.1.2 Automorphisms of the disk

It is also good to know the classification of automorphisms of the disk and upper half-plane. We define an *automorphism* to be a conformal map from an open set to itself.

**Example 4.2:** Show that the mappings

$$z \longmapsto \frac{\alpha - z}{1 - \overline{\alpha} z},$$

where $|\alpha| < 1$, are automorphisms of the unit disk $\mathbb{D}$.

*Solution:* First note that any such map that is holomorphic in $\mathbb{D}$, since $|\alpha| < 1$. On the boundary $|z| = 1$, we compute

$$\frac{\alpha - e^{i\theta}}{1 - \overline{\alpha} e^{i\theta}} = e^{-i\theta} \frac{w}{\overline{w}},$$

where $w = \alpha - e^{i\theta}$. Thus, any such map has unit norm on $\partial\mathbb{D}$, so by the maximum modulus principle we conclude that

$$z \longmapsto \frac{\alpha - z}{1 - \overline{\alpha} z},$$

is a mapping $\mathbb{D} \to \mathbb{D}$. A standard computation shows that this mapping is its own inverse.

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We also note that these mappings, since they are idempotent, interchange $z = 0$ and $z = \alpha$.

**Theorem 4.2 (Automorphisms of the Disk)**

All automorphisms of the disk $\mathbb{D}$ are of the form

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

for some $\theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$.

The proof follows from studying the function

$$g(z) = \left( f \circ \frac{\alpha - (\cdot)}{1 - \overline{\alpha}(\cdot)} \right)(z),$$

where $\alpha$ is chosen such that $f(\alpha) = 0$. This choice ensures that $g(0) = 0$, so that the Schwarz lemma may be applied to $g^{-1}$. The details are left as an exercise. An immediate corollary of this theorem is that the automorphisms of $\mathbb{D}$ that fix the origin are just the rotations.

**Example 4.3 (January 2015, Problem 1):** Find all conformal mappings of the domain for which $z$ satisfies both $|\arg(z)| < \pi$ and $|z| < 1$ onto the unit disk such that the image of $z = 1/2$ is zero.

*Solution:* Let $D$ denote the domain in the question. Then $D$ describes the unit disk without the negative real axis. In particular, $f(z) = \sqrt{z}$, taken with the principal branch, is conformal on $D$ and maps it to the right half-disk. Multiplying by $i$, we get a conformal map from $D$ to the upper half-disk.

Now we want to unfold the right half-disk to the upper half-plane, from which we can use the usual transformation

$$F(z) = \frac{i - z}{i + z}$$

to map to the disk. We apply the Joukowski transformation

$$J(z) = -\left(z + \frac{1}{z}\right)$$

to map the upper half-disk to the upper half-plane, so that $F \circ J \circ (if)$ is a conformal map from $D$ to $\mathbb{D}$. Checking the image of $z = 1/2$ under $J \circ (if)$, we see that

$$(J \circ (if))\left(\frac{1}{2}\right) = \frac{i}{\sqrt{2}},$$

so we need to slightly modify $F$ to ensure that $z = 1/2$ maps to the origin. The straightforward choice is

$$\tilde{F}(z) = \frac{i - \sqrt{2}z}{i + \sqrt{2}z},$$

so one such conformal mapping of $D$ to $\mathbb{D}$ is $g(z) = \tilde{F} \circ J \circ (if)$.

To find all such conformal maps $D \to \mathbb{D}$, let us consider any arbitrary $h$ satisfying the hypotheses of the question. Then $H \colon \mathbb{D} \to \mathbb{D}$ defined by $H(z) = h \circ g^{-1}$ is conformal and maps the origin to itself. Hence, $H$ is a rotation by the classification of automorphisms of the disk, so in fact

$$h(z) = e^{i\theta} g(z)$$

for some $\theta \in \mathbb{R}$.

■

## 4.2 Riemann Mapping Theorem

The Riemann mapping theorem is probably overkill for most of the questions that appear on the exam but is nonetheless good to know.

**Theorem 4.3 (Riemann mapping theorem)**

Let $D$ be a domain which is a proper subset of $\mathbb{C}$. For any $z_0 \in D$, there exists a unique conformal map $F \colon D \to \mathbb{D}$ such that

$$F(z_0) = 0, \quad F'(z_0) > 0.$$

The theorem can be useful if all you need is some mapping into the disk but do not necessarily care about the exact form of the mapping. The next example illustrates this.

**Example 4.4 (January 2011, Problem 3 (again)):** The function $f$ is analytic in the whole plane and have positive imaginary part. What can it be? What if all you know is that the imaginary part of $f$ tends to 0 at $\infty$?

*Solution:* We are given that $f$ is a mapping $\mathbb{C} \to \mathbb{H}$. By the Riemann mapping theorem, there is a conformal mapping $g\colon \mathbb{H} \to \mathbb{D}$. Then $g \circ f$ is entire and bounded, hence constant by Liouville's theorem, so that $f = g^{-1} \circ g \circ f$ is also constant.

If the imaginary part of $f$ tends to 0 at infinity, then $f$ is mapped into some horizontal strip $D = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < M\}$ for some $M > 0$. Choosing a conformal mapping $h\colon D \to \mathbb{D}$, the same argument above shows that $f$ is also constant in this case.

■

**Example 4.5:** Does there exist a holomorphic surjection $f\colon \mathbb{D} \to \mathbb{C}$.

*Solution:* By the Riemann mapping theorem, there is a conformal map $f\colon \mathbb{D} \to D$ where $D$ is the upper half-plane shifted down by one unit. Composing $f$ with $g(z) = z^2$ then gives the desired surjection.

■

## 4.3 More Examples/Exercises for Conformal Maps

**Example 4.6 (January 1990, Problem 4):** Find a conformal map $w = f(z)$ from the wedge $0 < \arg(z) < \alpha\pi$, where $0 < \alpha < 1$, onto the unit disk.

*Solution:* Any $z$ in the wedge $D_\alpha$ can be written as $z = re^{i\theta}$ where $0 < \theta < \alpha\pi$. Hence, the mapping $g(z) = z^{1/\alpha}$ maps the wedge to the upper half-plane. Now we have the mapping

$$F(z) = \frac{i - z}{i + z}$$

from the half-plane to the unit disk, so that $F \circ g$ maps the wedge $D_\alpha$ to the unit disk $\mathbb{D}$.

■

**Example 4.7 (September 2024, Problem 2):** Construct a conformal mapping from the domain

$$D = \mathbb{H} \backslash \{z = e^{i\theta} : \theta \in (0, \pi/2]\}$$

to $\mathbb{H}$.

*Solution:* We will write the map as a composition of conformal maps. First, we map $D$ under the linear fractional transformation

$$f_1(z) = \frac{z - 1}{z + 1}.$$

Under this transformation, the slit $e^{i\theta}$ becomes

$$\frac{e^{i\theta}-1}{e^{i\theta}+1}=\frac{e^{i\theta}-1}{e^{i\theta}+1}\cdot\frac{e^{-i\theta}+1}{e^{-i\theta}+1}=i\frac{\sin(\theta)}{1+\cos(\theta)}.$$

Thus, $f_1$ maps $D$ to the domain $D_1=\mathbb{C}\backslash(0,i]$, the complex plane without the segment from 0 to $i$ on the imaginary axis.

Next, consider the image of $D_1$ under the map $f_2(z)=z^2$, which doubles the argument of every $z\in D_1$. Since the real-axis is excluded, $f_2(D_1)$ is the domain $D_2=\mathbb{C}\backslash[-1,\infty)$. Now let $f_3(z)=z+1$ to map $D_2$ to $D_3=\mathbb{C}\backslash[0,\infty)$, which is then mapped to $\mathbb{H}$ by the square-root map $f_4(z)=\sqrt{z}$. Thus, the final map is

$$f(z)=(f_4\circ f_3\circ f_2\circ f_1)(z)=\sqrt{\left(\frac{z-1}{z+1}\right)^2+1}.$$

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**Exercise 4.1 (September 2021, Problem 2):** Construct a conformal map that sends the half-strip

$$S=\{a+ib:0<a<\pi,\ b>0\}$$

to the upper half-plane. You may express your map as a composition of functions.

**Exercise 4.2 (January 2022, Problem 5):** Construct a one-to-one conformal map from $\mathbb{H}$ to the region of $\mathbb{C}$ below the parabola $y=9x^2$, i.e. $\{z=x+iy:y<9x^2\}$. Express your answer as an analytic function of $z$.

## 4.4 Practice Exams

To conclude, let's go through the solutions to some recent exams question-by-question.

### 4.4.1 September 2025 Solutions

Note that this exam specified to choose 5 of the following 6 questions to solve. I'll provide my solutions to all 6 problems but don't worry if the exam seems a bit long with all 6 of them.

**1.** Let $f(z)$ be an entire function which is not a constant function. Prove that for any complex number $c$ there is an infinite sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} f(z_n)=c.$$

*Solution:* We previously saw solutions to this using Little Picard and another using Casorati-Weierstrass. We present yet another solution that only requires Liouville's theorem. Suppose instead that the range of $f$ is not dense in $\mathbb{C}$. Then we may choose a ball $B(w, \epsilon)$ such that $f$ never attains a value in this ball. Now define the auxiliary function

$$g(z) = \frac{1}{f(z) - w}.$$

Since $f(z) \neq w$ for all $z \in \mathbb{C}$, the function $g$ is entire. Moreover, $|f(z) - w| \geq \epsilon$ for all $z \in \mathbb{C}$, so that $g$ is bounded. Hence, we may apply Liouville to conclude that $g$ is constant, which then implies that $f$ is constant, a contradiction.

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**2.** Let $f(z)$ be an entire function such that $f(x + 2\pi) = f(x)$ for all real $x$. Is it then true that

$$f(z + 2\pi) = f(z)$$

for all complex $z$? If yes, prove it. If no, give a counterexample.

*Solution:* Define the function $g(z) = f(z + 2\pi) - f(z)$. Then $g$ vanishes on the real line, so by the identity theorem, $g$ vanishes on all of $z$. Thus, the statement is true.

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**3.** Consider the complex function

$$f(z) = \frac{1}{z(z+1)(z+2)}.$$

Obtain the Laurent series expansion of the function in the region $|z| > 2$. *Note: the exam does the annulus $1 < |z| < 2$, but we have already used this as an example.*

*Solution:* By partial fractions, we have that

$$f(z) = \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{2(2-z)}.$$

In order for the series to converge, we must rewrite this as

$$\begin{aligned} f(z) &= \frac{1}{2z} + \left(-\frac{1}{z}\right)\left(\frac{1}{1 - 1/z}\right) + \left(\frac{1}{4z}\right)\left(\frac{1}{1 - 1/(2z)}\right) \\ &= \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \frac{1}{z^n} \\ &= -\frac{1}{4z} + \sum_{n=2}^{\infty} \left(\frac{1}{2^{n+1}} - 1\right) \frac{1}{z^n}. \end{aligned}$$

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**4.** Use the method of residues to compute the real integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2 + a^2)^2}\, dx,$$

where $a > 0$.

*Solution:* Let $C_R$ be the arc of the circle of radius $R$ with argument from 0 to $\pi$ and let $\Gamma_R = [-R, R]$. Define $C = C_R \cup \Gamma_R$, taken to be positively oriented. Let

$$f(z) = \frac{e^{iz}}{(z^2 + a^2)^2} = \frac{e^{iz}}{(z - (ia))^2 (z + (ia))^2},$$

and observe that $f$ has double poles at $z = \pm ia$. Computing the residue at $z = ia$, we find that

$$2\pi i \operatorname{Res}_{z=ia} f(z) = 2\pi i \frac{ie^{i(ia)}(2ia)^2 - 2e^{i(ia)}(2ia)}{(2ia)^4} = 2\pi i \frac{ie^{-a}(2ia) - 2e^{-a}}{(2ia)^3} = \pi \left( \frac{a+1}{2a^3} \right) e^{-a}$$

It remains to show that the integral of $f$ over $C_R$ vanishes as $R \to \infty$. Indeed,

$$\begin{aligned} \left| \int_{C_R} \frac{e^{iz}}{(z^2+a^2)^2} \, dz \right| &\leq \int_{C_R} \left| \frac{e^{iz}}{(z^2+a^2)^2} \right| dz \\ &\leq \int_{C_R} \frac{1}{(R^2 - a^2)^2} \, dz \\ &\leq \frac{\pi R}{(R^2 - a^2)^2}, \end{aligned}$$

which vanishes in the limit $R \to \infty$. Here, we have used the reverse triangle inequality and that $e^{-y} \leq 1$ for $y \geq 0$. Thus, we find that

$$I = \pi \left( \frac{a+1}{2a^3} \right) e^{-a}.$$

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**5.** Let $f(t)$ be a continuous and bounded function for $t \geq 0$ and define

$$g(z) = \int_0^\infty f(t) e^{-zt} \, dt, \quad \operatorname{Re}(z) > 0,$$

known as the Laplace transform of $f(t)$. Show that $g(z)$ is holomorphic in the domain $\operatorname{Re}(z) > 0$.

*Solution:* Consider the truncated integrals

$$g_n(z) = \int_0^n f(t) e^{-zt} \, dt.$$

Then, since the integration domain is finite and the integrand is continuous in $t$, one can differentiate under the integrand sign to conclude that $g_n$ is entire. It suffices now, due to a consequence of Morera's theorem previously discussed (Example 2.4), to show that $g_n \to g$ on compact subsets of $\operatorname{Re}(z) > 0$. Let $D$ be any compact subset of $\operatorname{Re}(z) > 0$ and let

$$K = \min_{z \in D} \operatorname{Re}(z) > 0.$$

Moreover, since $f$ is bounded, then there is $M > 0$ such that $|f(t)| \leq M$ for all $t \geq 0$. Now,

$$|g(z) - g_n(z)| \leq \int_n^\infty |f(t)|\left|e^{-zt}\right| \, dt \leq \int_n^\infty Me^{-t\mathrm{Re}(z)} \, dt \leq \int_n^\infty Me^{-Kt} \, dt = \frac{M}{K}e^{-Kn},$$

which tends to 0 in the limit $n \to \infty$. Thus, $g_n \to g$ on any compact subset of $\mathrm{Re}(z) > 0$, so that $g$ is holomorphic on that domain.

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**6.** Use the argument principle to determine the number of zeros of the function

$$f(z) = z^4 + 6z + 3$$

inside the circle $|z| = 2$.

*Solution:* Note that Rouché's theorem is a consequence of the argument principle, so we will just apply Rouché's theorem to solve this question. If you have time on the exam, you can re-derive Rouché from the argument principle.

On $|z| = 2$, the largest term in the polynomial $f$ is

$$2^4 = 16 > 15 = 6(2) + 3.$$

Therefore, Rouché's theorem says that there are four zeros of $f$ in the circle $|z| = 2$ (counting multiplicity).

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### 4.4.2 January 2022 Solutions

**1.** Let

$$f(z) = \frac{3}{z^2 - z - 2}.$$

Find the Laurent series for $f$ in each annulus centered on the origin which is relevant for this function. In which annulus or annuli does $f$ have primitives (a.k.a. antiderivatives)?

*Solution:* We first write

$$f(z) = \frac{3}{(z-2)(z+1)} = \frac{1}{z-2} - \frac{1}{z+1}.$$

Since $f$ has poles at $z = -1$ and $z = 2$, the three regions we can study are $|z| < 1$, $1 < |z| < 2$, and $|z >> 2|$. Let's just find the Laurent series in the annulus $1 < |z| < 2$. We write

$$\begin{aligned} f(z) = \frac{1}{z-2} - \frac{1}{z+1} &= -\frac{1}{2} \cdot \frac{1}{1-(z/2)} - \frac{1}{z} \cdot \frac{1}{1-(-1/z)} \\ &= -\sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} - \sum_{k=1}^{\infty} (-1)^{k-1} z^{-k}. \end{aligned}$$

By the antiderivative theorem, $f$ has a primitive in a region if $\int_C f = 0$ for any simple closed contour $C$ in the region. However, Cauchy's theorem says that $\int_C f$ is proportional to the sum of the residues of $f$ at the singular points. For the annulus we consider above, the residue is $-1$, so $f$ does not admit an antiderivative in this region.

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**2.**

(a) Let $\gamma$ be the ellipse $x^2/2+y^2 = 1$, traversed in the counterclockwise direction. Compute

$$\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz.$$

(b) Let $a > 0$ be a strictly positive real number. Compute

$$I_a = \int_0^\infty \frac{\cos(\ln(x))}{(x+a)^2}\, dx.$$

*Solution:*

(a) The integrand has a pole of order 3 at $z = 0$. By Cauchy's theorem,

$$\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz = 2\pi i \operatorname{Res}_{z=0} f(z) = \frac{2\pi i}{3},$$

where the residue is readily found by expanding the integrand as a Laurent series.

(b) This one is pretty annoying. Let's use a "key hole" contour, defined as follows. Let $C_R$ and $C_\epsilon$ be circles of radius $R$ and $\epsilon$ centered at the origin, respectively. Let $L_\epsilon^+$ and $L_\epsilon^-$ be lines above and below the real axis, respectively, of distance $\epsilon$ away from the real axis, each running from $z = 0$ to $z = R$. Now let $C = C_R \cup C_\epsilon \cup L_\epsilon^+ \cup L_\epsilon^-$ be positively oriented. We will evaluate the integral of the function

$$f(z) = \frac{z^i}{(z+a)^2} = \frac{e^{i\log(z)}}{(z+a)^2},$$

where the branch cut is the positive real axis.

Let's first show that the circle integrals vanish in the limits $R \to \infty$ and $\epsilon \to 0$. First, note that

$$\left|z^i\right| = e^{-\theta},$$

so that $|z^i| \leq 1$. Thus,

$$\left|\int_{C_R} f\right| \leq C\frac{R}{(R-a)^2} \to 0$$

as $R \to \infty$. Similarly,

$$\left|\int_{C_\epsilon} f\right| \leq C\frac{\epsilon}{(a-\epsilon)^2} \to 0$$

as $\epsilon \to 0$.

Now let's compute the integrals along $L_\epsilon^\pm$ in the limits $\epsilon \to 0$ and $R \to \infty$. We parametrize $L_\epsilon^+$ as $z = x + i\epsilon$ where $x \in [0, \sqrt{R^2 - \epsilon^2}]$. Then

$$\int_{L_\epsilon^+} f(z)\ dz = \int_0^{\sqrt{R^2-\epsilon^2}} \frac{(x+i\epsilon)^i}{(x+i\epsilon+a)^2}\ dx,$$

which by the dominated convergence theorem becomes

$$\int_0^R \frac{x^i}{(x+a)^2}\ dx.$$

Likewise,

$$\int_{L_\epsilon^-} f(z)\ dz = -e^{-2\pi} \int_0^R \frac{x^i}{(x+a)^2}\ dx.$$

Finally, we compute the residue at $x = -a$. A standard computation gives us

$$\operatorname{Res}_{z=-a} f(z) = -\frac{ie^{-\pi}}{a} e^{i\log(a)}.$$

Letting

$$J = \int_0^\infty \frac{x^i}{(x+a)^2},$$

the residue theorem gives us that

$$(1 - e^{-2\pi})J = 2\pi i \operatorname{Res}_{z=-a} f(z) = -(2\pi i)\frac{ie^{-\pi}}{a} e^{i\log(a)}.$$

After rearranging, we conclude that

$$I_a = \operatorname{Re}(J) = \frac{2\pi e^{-\pi}\cos(\log(a))}{a(1 - e^{-2\pi})}.$$

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**3.**

(a) Let $f$ be an entire function with power series expansion $f(z) = \sum_{n=0}^\infty c_n z^n$. Assume that $f$ is purely real on the real axis. What can you say about the coefficients $c_n$?

(b) Determine the general form of harmonic functions on $\mathbb{R}^2$ which vanish on the real axis.

(c) Derive, using the previous question, the general form of the real harmonic polynomials $P(x, y)$ with $P(x, 0) = x^3 - 2x + 1$.

*Solution:*

(a) We claim that all the coefficients $c_n$ are real. Indeed, define

$$g(z) = f(z) - \overline{f(\overline{z})}.$$

Then $g$ vanishes on the real axis, so by the identity theorem $g$ is identically zero. Therefore, $c_n = \overline{c_n}$ for all non-negative integers $n$, so that each $c_n$ is real.

(b) Any harmonic function $v$ on $\mathbb{R}^2$ is the imaginary part of an entire function. To this end, write $f(z) = u + iv$ where $u$ is the harmonic conjugate of $v$. Since $f$ is entire, we can expand $f$ as a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Under the assumptions of the question, $v$ vanishes on the real axis, so that $f$ is purely real on the real axis. Therefore, each $c_n$ is real (by part (a)). Thus, $v$ is of the form

$$v(x,y) = \mathrm{Im}(f(z)) = \sum_{n=0}^{\infty} c_n \mathrm{Im}(z^n).$$

(c) Since $P(x,y)$ is a harmonic polynomial, it is the imaginary part of a complex polynomial. Thus,

$$P(x,y) = \mathrm{Im}\left(\sum_{n=0}^{N} c_n z^n\right).$$

The data $P(x,0) = x^3 - 2x + 1$ then gives the relation

$$\sum_{n=0}^{N} \mathrm{Im}(c_n) x^n = x^3 - 2x + 1,$$

from which we conclude $\mathrm{Im}(c_0) = 1$, $\mathrm{Im}(c_1) = -2$, and $\mathrm{Im}(c_3) = 1$, and the remaining $c_n$ are all real. In particular,

$$P(x,y) - \mathrm{Im}(c_0 + c_1 z + c_3 z^3)$$

is a real harmonic polynomial that vanishes on the real axis. Therfore, by part (b),

$$P(x,y) - \mathrm{Im}(c_0 + c_1 z + c_3 z^3) = \mathrm{Im}\left(\sum_{n=0}^{N} d_n z^n\right)$$

for some real coefficients $d_n$. In particular,

$$P(x,y) = \mathrm{Im}\left(\sum_{n=0}^{N} b_n z^n\right)$$

where all $b_n$ are real, except for $b_0, b_1, b_3$, which satisfy

$$\mathrm{Im}(b_0) = 1, \quad \mathrm{Im}(b_1) = -2, \quad \mathrm{Im}(b_3) = 1.$$

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**4.** Let $D$ be the unit disk centered at $z = 0$ of unit radius, and let $\overline{D}$ be the closure of $D$. Assume $f\colon \overline{D} \to \mathbb{C}$ is continuous on $\overline{D}$ and analytic on $D$. Suppose further that $f$ is non-constant and such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$.

(a) Show that $f$ has a zero in $D$.

(b) Show that $f(D) = D$.

*Solution:*

(a) Suppose instead $f$ is non-zero in $D$. Then the function $1/f$ is analytic in $D$ with $|1/f| > 1$, since the maximum modulus principle says that $|f| < 1$ in $D$. However, $|1/f| = 1$ on the unit circle, a contradiction.

(b)

*Solution 1 (MMP):* The maximum modulus principle immediately gives that $f(D) \subseteq D$. Now suppose $w \in D \backslash f(D)$ and consider the auxiliary function

$$g(z) = \frac{1}{f(z) - w}.$$

Then $g(z)$ is holomorphic in $D$, and the maximum modulus principle says that

$$\frac{1}{|f(z) - w|} \leq \sup_{|z|=1} \frac{1}{|f(z) - w|} \leq \sup_{|z|=1} \frac{1}{||f(z)| - |w||} = \frac{1}{1 - |w|}.$$

Thus, $1 - |w| \leq |f(z) - w|$, so the disk of radius $1 - |w|$ centered at $w$ does not intersect $f(D)$. In particular, $D \backslash f(D)$ is open. However, the open mapping theorem says that $f(D)$ is open, a contradiction since $D$ is connected.

*Solution 2 (Rouché):* Given $w \in D$, it suffices to show that $g(z) = f(z) - w$ has a root in $D$. We have that the constant function $-w$ satisfies $|-w| < 1 = |f(z)|$ on the unit circle. Thus, by Rouché, $f$ and $g$ have the same number of roots in $D$, which is at least one by part (a).

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**5.** Let $\mathbb{H}$ be the upper half-plane. Construct a one-to-one conformal map from $\mathbb{H}$ to the region below the parabola $y = 9x^2$, i.e. $\{z = x + iy \in \mathbb{C} : y < 9x^2\}$. Please express your answer as an explicit function of $z$.

*Solution:* We want the boundary of $\mathbb{H}$, the real axis, to map to the parabola $v = 9u^2$ in the $w$-plane. We know that the map $z \mapsto z^2$ maps lines to parabolas or rays, so let us look for a map of the form

$$f(z) = c + bz + az^2.$$

Since $v = 9u^2$ intersects the origin, we can set $c = 0$ so that the origin maps to itself. Restricting to the real line, $f$ becomes

$$f(x) = bx + ax^2,$$

and the imaginary part of $f$ must satisfy $v = 9u^2$. This quadratic dependence of $v = \text{Im}(f)$ on the $u = \text{Re}(f)$ suggests that $a$ should be purely imaginary. For simplicity, we set $a = 9i$. Then

$$f(x) = bx + (9i)x^2.$$

Letting $b = -1$, we find that $u = x$ and $v = i(9u^2)$, so our candidate function is

$$f(z) = -z + (9i)z^2.$$

Note that $b = 1$ would not work, as then injectivity would fail. Computing the derivative, we see that

$$f'(z) = -1 + 18iz,$$

which vanishes at $z = -i/18 \notin \mathbb{H}$. Thus, $f$ is conformal on $\mathbb{H}$. It remains to check that $\mathbb{H}$ folds under the parabola. Indeed, checking the point $z = i$, we compute $f(i) = -10i$, so by continuity $f$ maps $\mathbb{H}$ under the parabola $v = 9u^2$.

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# References

[1] J. W. Brown and R. V. Churchill. *Complex Variables and Applications.* McGraw-Hill Higher Education, Boston, MA, eighth edition, 2009.

[2] E. Stein and R. Shakarchi. *Complex Analysis.* Princeton lectures in analysis. Princeton University Press, 2010.