Honors Analysis I

Recitation 1 Notes

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1. Introduction

Welcome to Honors Analysis I! I'll post my notes for each recitation on my website:

graysonrdavis.github.io

General Information:

- TA: Grayson Davis
- Office hours: CIWW 608,
  - ◇ Tuesdays, 1:00pm - 2:00pm.
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The structure of the recitation will roughly be as follows:

(1) A 15 - 25 minute quiz (length dependent on question difficulty). The question on the quiz will come directly from your homework.

(2) Quiz solutions.

(3) Other problem solutions and concept review.

(4) Q&A.

If you have any problem solutions or concepts you want me to cover in recitation, please email me by the Wednesday before recitation.

2. Quiz solutions

1. **(Question 4)** Given a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$|a_{n+1} - a_n| \leq \lambda |a_n - a_{n-1}|$$

for $n \geq 2$, with $0 < \lambda < 1$. Show that the sequence converges.

Proof. We show the sequence is Cauchy, which implies it is convergent by completeness of the real numbers. Assume without loss of generality that $a_1 \neq a_2$.

First, we claim by induction that

$$|a_{n+1} - a_n| \leq \lambda^{n-1} |a_2 - a_1|.$$

The base case is immediate. For $n > 2$, we have by induction,

$$|a_{n+1} - a_n| \leq \lambda |a_n - a_{n-1}| \leq \lambda^{n-1} |a_2 - a_1|.$$

Fix $\epsilon > 0$. Given any $m, n \in \mathbb{N}$ with $m > n$, we have by the triangle inequality

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + \ldots + |a_{n+1} - a_n| \\
&\leq |a_2 - a_1| \sum_{k=n}^{m-1} \lambda^{k-1} \\
&\leq |a_2 - a_1| \lambda^{n-1} \frac{1 - \lambda^{m-n}}{1 - \lambda} \\
&\leq |a_2 - a_1| \frac{\lambda^{n-1}}{1 - \lambda},
\end{aligned}$$

where we have used the geometric series formula and that $\lambda < 1$. Choose $N \in \mathbb{N}$ sufficiently large such that

$$\lambda^{n-1} < \frac{(1 - \lambda)}{|a_2 - a_1|} \epsilon.$$

Thus, for $m > n \geq N$, we obtain $|a_m - a_n| < \epsilon$. Hence $\{a_n\}_n$ is Cauchy. □

2. **(Question 6)** Prove that if $\{a_n\}_n$ converges to L then $\{\sigma_n\}_n$ converges to L, where

$$\sigma_n = \frac{a_1 + \ldots + a_n}{n}.$$

Proof. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - L| < \frac{\epsilon}{2}.$$

Given any such $n \geq N$, we have by the triangle inequality

$$\begin{aligned}
|\sigma_n - L| \leq \frac{1}{n} \sum_{k=1}^{n} |a_k - L| &\leq \frac{1}{n} \left(\sum_{k=1}^{N-1} |a_k - L| \right) + \frac{1}{n} \left(\sum_{k=N}^{n} |a_k - L| \right) \\
&\leq \frac{1}{n} \left(\sum_{k=1}^{N-1} |a_k - L| \right) + \frac{n - N + 1}{n} \cdot \frac{\epsilon}{2}.
\end{aligned}$$

Since N is fixed, we can choose a natural number $M \geq N$ such that

$$\frac{1}{M} \left(\sum_{k=1}^{N-1} |a_k - L| \right) < \frac{\epsilon}{2}.$$

Thus, taking $n \geq M$, we obtain

$$|\sigma_n - L| \leq \frac{1}{n} \left(\sum_{k=1}^{N-1} |a_k - L| \right) + \frac{n - N + 1}{n} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. □

3. Other Problem Solutions

3. **(Question 7)** Give an example of a sequence $\{a_n\}_n$ such that the sequence

$$\sigma_n = \frac{a_1 + \ldots + a_n}{n}$$

converges but $\{a_n\}_n$ diverges.

Proof. Define $a_{2k} = -1$, $a_{2k-1} = 1$ for $k \geq 1$. Then $\sigma_{2k} = 0$ and $\sigma_{2k-1} = 1/n$. Hence $\sigma_n \to 0$ as $n \to \infty$. However $\{a_n\}_n$ does not have a limit. □

4. **(Question 3)** True or false:

   (a) If a sequence $\{a_n\}_n$ is positive and unbounded, then $\lim_{n\to\infty} a_n = +\infty$.

   Proof. False. Consider the sequence $a_{2k} = 1$ and $a_{2k+1} = k$. □

   (b) If a sequence $\{a_n\}_n$ is positive, bounded, and $\lim_{n\to\infty}(a_{n+1}-a_n) = 0$, then $\lim_{n\to\infty} a_n$ exists.

   Proof. False. Consider the sequence $\{a_n\}_n$ defined by
$$a_n = 2 + \cos\big(\pi\sqrt{n}\big).$$
   Then $\{a_n\}_n$ is positive and bounded, since $|\cos(x)| \leq 1$ for every $x \in \mathbb{R}$. Moreover, by the mean-value theorem
$$\left|\cos\left(\pi\sqrt{n+1}\right) - \cos\big(\pi\sqrt{n}\big)\right| \leq \pi(\sqrt{n+1} - \sqrt{n}) \to 0$$
   as $n \to \infty$. However,
$$2 + \cos\left(\pi(\sqrt{(2n+1)^2})\right) = 1, \quad 2 + \cos\left(\pi(\sqrt{(2n)^2})\right) = 3,$$
   so the limit does not exist. □

   (c) If a sequence $\{a_n\}_n$ is positive, bounded, and $\lim_{n\to\infty} a_{n+1}/a_n = 1$, then the sequence $\{a_n\}_n$ converges.

   Proof. False. Consider the sequence $\{a_n\}_n$ from part (b). We have
$$\left|\frac{a_{n+1}}{a_n} - 1\right| = \left|\frac{\cos\big(\pi\sqrt{n+1}\big) - \cos(\pi\sqrt{n})}{2 + \cos(\pi\sqrt{n})}\right| \leq |a_{n+1} - a_n| \to 0$$
   as $n \to \infty$. Yet the sequence $\{a_n\}_n$ does not converge. □

   (d) If a sequence $\{a_n\}_n$ has the property that $\lim_{n\to\infty}(a_{n+p} - a_n) = 0$ for any fixed $p \in \mathbb{N}$, then the sequence $\{a_n\}_n$ converges.

   Proof. False. The same counterexample as in (b) and (c) applies here. □

5. **(Question 5)** Show that if $\{a_n\}_n$ and $\{b_n\}_n$ are bounded above, then
$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$
   Exercise: State and prove an analogous statement for $\liminf$.

   Proof. Given $n \in \mathbb{N}$, observe that, since sup is the least upper bound,
$$\sup_{k\geq n}(a_n + b_n) \leq \sup_{k\geq n} a_n + \sup_{k\geq n} b_n.$$
   Therefore, we may take $n \to \infty$ to obtain the desired inequality. □