Honors Analysis I

Recitation 10 Notes

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1. Arzelà-Ascoli

Today we will prove the Arzelà-Ascoli theorem and work through some examples applying the theorem. Recall that a subset of $\mathbb{R}^d$ (or any finite-dimensional space) is compact if and only if it is closed and bounded. We showed in Recitation 8 that this result characterizes finite-dimensional spaces: a normed linear space is finite dimensional if and only if the closed unit ball in the space is compact. The Arzelà-Ascoli theorem gives a similar statement for the space of continuous functions on a compact metric space. Before stating the theorem, let's review some basic definitions.

Let $\mathcal{F} \subseteq C(X)$ be a subset of continuous functions on some metric space X equipped with the supremum norm $\|\cdot\|_\infty$. Then

- $\mathcal{F}$ is *relatively compact* in $C(X)$ if $\overline{\mathcal{F}}$ is compact,
- $\mathcal{F}$ is *bounded* in $C(X)$ if there is $M > 0$ such that $\|f\|_\infty \leq M < \infty$ for all $f \in \mathcal{F}$,
- $\mathcal{F}$ is *totally bounded* if for every $\epsilon > 0$ there are finitely many balls $B_\epsilon(f_1), \ldots, B_\epsilon(f_n)$ all of radius ϵ that cover $\mathcal{F}$.
- $\mathcal{F}$ is equicontinuous at $x \in X$ if given $\epsilon > 0$ there is $\delta_x > 0$ such that for any $y \in X$ with $d(x, y) < \delta_x$ and any $f \in \mathcal{F}$ it holds that $|f(x) - f(y)| < \epsilon$.
- $\mathcal{F}$ is uniformly equicontinuous at $x \in X$ if given $\epsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$ and any $f \in \mathcal{F}$ it holds that $|f(x) - f(y)| < \epsilon$.

Equicontinuity is a very strong statement about the family of functions $\mathcal{F}$, implying not only that each function in the family is uniformly continuous, but additionally insisting that the constants can be chosen to be uniform over the entire family $\mathcal{F}$. Note that if X is compact, then equicontinuity is equivalent to uniform equicontinuity:

Lemma 1.1. *Let X be a compact metric space and $\mathcal{F} \subseteq C(X)$ a collection of real-valued continuous functions on X. Then $\mathcal{F}$ is equicontinuous if and only if $\mathcal{F}$ is uniformly equicontinuous.*

We leave the proof as an exercise.

Our goal is to prove the following:

Theorem 1.1 (Arzelà-Ascoli)**.** *Let X be a compact metric space and consider the space of real-valued continuous functions $C(X)$ on X with the supremum norm. Let $\mathcal{F} \subseteq C(X)$. Then the following are equivalent:*

(i) $\mathcal{F}$ is relatively compact in $C(X)$,

(ii) $\mathcal{F}$ is bounded and equicontinuous.

As a corollary, we obtain "Theorem 8.8" from the lecture notes:

Corollary 1.1. *Let X be a compact metric space and let $\{f_n\}_n \subseteq C(X)$ be a equicontinuous and uniformly bounded sequence. Then $\{f_n\}_n$ has a uniformly convergent subsequence.*

Proof. Since $\{f_n\}_n$ is equicontinuous and uniformly bounded, the Arzelà-Ascoli theorem says that $\overline{\{f_n\}_n}$ is compact. Hence there is a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ that converges to some $f \in \overline{\{f_n\}_n}$ in the supremum norm, so that the convergence is uniform. □

We will prove Theorem 1.1 through a sequence of intermediate lemmas. First, we show that $\mathcal{F}$ is bounded and equicontinuous if $\mathcal{F}$ is relatively compact.

First note that if $\mathcal{F}$ is relatively compact, then $\overline{\mathcal{F}}$ is bounded, and hence $\mathcal{F}$ is bounded. To show equicontinuity, we apply the following lemma:

Lemma 1.2. *If $\mathcal{F}$ is totally bounded, then $\mathcal{F}$ is equicontinuous.*

Indeed, any open cover of $\overline{F}$ admits a finite subcover, so that $\mathcal{F}$ is totally bounded. Let's prove the lemma:

Proof. Fix $\epsilon > 0$. Since $\mathcal{F}$ is totally bounded, there are finitely many functions $f_1, \ldots, f_N \in C(X)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{N} B_{\frac{\epsilon}{3}}(f_i).$$

We note that the collection $\{f_1, \ldots, f_n\}$ is equicontinuous, as it is finite. Thus there is $\delta > 0$ such that for every $1 \leq i \leq N$, we have $|f_i(x) - f_i(y)| < \epsilon$ whenever $d(x, y) < \delta$. Now fix $f \in \mathcal{F}$ and choose f_i such that $f \in B_{\epsilon/3}(f_i)$. Then for any $x, y \in X$ with $d(x, y) < \delta$, it follows that

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \epsilon,$$

as desired. □

Thus, Lemma 1.2 implies the first direction of Theorem 1.1. For the second direction, we require:

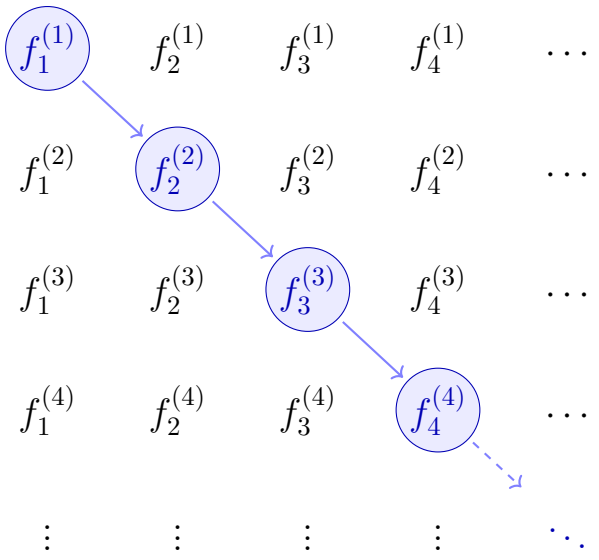
Lemma 1.3. *A compact metric space is separable (i.e. has a countable dense subset).*

We leave the proof of Lemma 1.3 as an exercise.

Now assume $\mathcal{F}$ is bounded and equicontinuous. We show that $\overline{F}$ is sequentially compact, and use that sequential compactness is equivalent to compactness in a metric space. Let $\{g_n\}_n \subseteq \overline{\mathcal{F}}$ be given.

By Lemma 1.3, there is a countable dense subset $\{x_k\}_k \subseteq X$. The idea of the proof is to test the sequence $\{g_n\}_n$ at this countable sequence of points and use denseness to extract a limit. Since $\{g_n\}_n$ lives in the closure of $\mathcal{F}$, first choose for every $n \in \mathbb{N}$ a function $f_n \in \mathcal{F}$ such that $\|f_n - g_n\|_\infty < 1/n$.

The diagonalization argument proceeds as follows. Since $\mathcal{F}$ is bounded, the sequence $\{f_n(x_1)\}_n$ is a bounded sequence in $\mathbb{R}$, and hence has a convergent subsequence $\{f_{n_k}(x_1)\}_k$. But then $\{f_{n_k}\}_k$ is also in the bounded set $\mathcal{F}$, so that $\{f_{n_k}(x_2)\}_k$ is a bounded sequence in $\mathbb{R}$, and hence has convergent subsequence. For simplicity of notation, define $f_k^{(1)} := f_{n_k}$, and let $\{f_k^{(2)}\}_k$ denote the subsequence of $\{f_k^{(1)}\}_k$ that converges at x_2 (and also x_1, by definition of the $f_k^{(1)}$). Continuing inductively, given $\{f_k^{(j)}\}_k$ a subsequence that converges at each $x_1, \ldots, x_j$ we again use boundedness of $\mathcal{F}$ to find a subsequence $\{f_k^{(j+1)}\}_k$ that converges at each $x_1, \ldots, x_j, x_{j+1}$.



Now define the diagonal sequence by $h_k = f_k^{(k)}$. We claim that h_k converges at every x_j. Indeed, if $k \geq j$, then each $h_k = f_k^{(k)}$ belongs to the previous row j, hence $\{h_k(x_j)\}_{k \geq j}$ converges. To show that h_k converges everywhere on X, we show that $\{h_k(x)\}_k$ is Cauchy for every $x \in X$.

Fix $\epsilon > 0$. By equicontinuity of $\mathcal{F}$ (and Lemma 1.1), there is $\delta > 0$ such that for any $f \in \mathcal{F}$ and any $x, y \in X$ satisfying $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon/3$. By denseness, we can choose $x_j \in \{x_k\}_k$ such that $d(x, x_j) < \delta$, and hence $|h_k(x) - h_k(x_j)| < \epsilon/3$ for every $k \in \mathbb{N}$. Thus, since $\{h_k(x_j)\}_k$ is Cauchy in $\mathbb{R}$, we obtain

$$|h_k(x) - h_l(x)| \leq |h_k(x) - h_k(x_j)| + |h_k(x_j) - h_l(x_j)| + |h_l(x_j) - h_l(x)| < \epsilon.$$

Hence $\{h_k(x)\}_k$ is Cauchy in $\mathbb{R}$, so we can define a function on X by $h(x) = \lim_{k \to \infty} h_k(x)$. We claim that the convergence is uniform, finishing the proof. Indeed, if $h_k \to h$ uniformly, then $h \in C(X)$, so that $h \in \overline{\mathcal{F}}$. Letting $h_k = f_{n_k}$ be the diagonal subsequence, we obtain

$$\|g_{n_k} - h\|_\infty \leq \|g_{n_k} - f_{n_k}\|_\infty + \|f_{n_k} - h\|_\infty \leq \frac{1}{n_k} + \|h_k - h\|_\infty \to 0$$

as $k \to \infty$. Thus we need only show that $h_k \to h$ uniformly to conclude the proof.

Passing to the limit in the definition of equicontinuity, we obtain that $|h(x) - h(y)| < \epsilon/3$ given any $x, y \in X$ with $d(x, y) < \delta$. Then compactness gives a finite number of balls $\{B_\delta(y_i)\}_{i=1..N}$ covering X. Fix $x \in X$ and choose i such that $x \in B_\delta(y_i)$. Then

$$|h_k(x) - h(x)| \leq |h_k(x) - h_k(y_i)| + |h_k(y_i) - h(y_i)| + |h(y_i) - h(x)| < \epsilon$$

for sufficiently large k. Therefore, $h_k \to h$ uniformly on X, concluding the proof.

2. Examples

Example 2.1. *Let $X = C([0, \pi])$ and $E = \{\sin(nt)\}_{n\in\mathbb{N}} \subseteq X$. Is E a relatively compact subset of X?*

Solution:

We claim that E is not relatively compact in X. Since E is bounded, it suffices (by Arzelà-Ascoli), to show that E is not equicontinuous. Let $\epsilon = 1$ and suppose $\delta > 0$ is given. Choose $N \in \mathbb{N}$ such that $\pi/(2N) < \delta$ and let $t = \pi/(2N)$. Then

$$|t - 0| = \frac{\pi}{2N} < \delta$$

while

$$|\sin(Nt) - \sin(0)| = 1 \geq \epsilon.$$

□

Another application is local existence of solutions to ordinary differential equations (ODEs).

Theorem 2.1 (Peano Existence Theorem)**.** *Let $a, b > 0$, $y_0 \in \mathbb{R}$, and suppose*

$$F\colon [0, a] \times [y_0 - b, y_0 + b] \to \mathbb{R}$$

is continuous. Then there exists $0 < T \leq a$ and a function $y \in C^1([0, T])$ satisfying

$$y'(t) = F(t, y(t)), \quad y(0) = y_0.$$

Proof. By the fundamental theorem of calculus, we want to find a continuous function y such that

$$y(t) = y_0 + \int_0^t F(s, y(s))\ ds.$$

The construction uses a piecewise linear approximation.

Since F is continuous on the compact rectangle $R \coloneqq [0, a] \times [y_0 - b, y_0 + b]$, the maximum $M = \max_{(t,x)\in R} |F(t, x)|$ exists and is finite. Choose $T = \min\{a, b/M\}$ and for every $n \in \mathbb{N}$, partition $[0, T]$ into n subintervals

$$I_k = [t_k, t_{k+1}] \coloneqq \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right].$$

Starting from y_0, we define a sequence of piecewise linear functions $\{y_n\}_{n\geq 0}$ by

$$y_n(t) = y_n(t_k) + \int_{t_k}^t F(t_k, y_n(t_k))\ ds, \quad t \in [t_k, t_{k+1}].$$

In particular, we are defining the y_n such that their difference quotient is F at the left endpoint. We may then expect that as the mesh size gets smaller (i.e. $n \to \infty$) that the y_n converge to some differentiable function y that satisfies the ODE. First we need to show that each y_n is well-defined on $[0, T]$. We prove that

$$|y_n(t_k) - y_0| \leq Mt_k \leq MT \leq b,$$

so that $(t_k, y_n(t_k)) \in R$. If $k = 0$, then $y_n(0) = y_0$ so the result is trivial. Now assume the estimate holds for $y_n(t_k)$. Then $F(t_k, y_n(t_k))$ is well-defined and satisfies $|F(t_k, y_n(t_k))| \leq M$, so that

$$|y_n(t_{k+1}) - y_0| \leq |y_n(t_k) - y_0| + |t_{k+1} - t_k||F(t_k, y_n(t_k))| \leq Mt_k + M(t_{k+1} - t_k) = Mt_{k+1}.$$

Thus, y_n is indeed well-defined on $[0, T]$.

We show that $\{y_n\}_n$ is bounded and equicontinuous. For boundedness, we see that

$$\|y_n\|_\infty \leq \|y_n - y_0\|_\infty + |y_0| \leq b + |y_0| < \infty.$$

For equicontinuity, fix $\epsilon > 0$ and let $t, s \in [0, T]$ be given. Then since F is uniformly bounded on each subinterval, we obtain

$$|y_n(t) - y_n(s)| \leq M|t - s|,$$

so choosing $\delta = \epsilon/M$ suffices. Thus, by Arzelà-Ascoli there is a subsequence $\{y_{n_k}\}_k$ of $\{y_n\}$ and $y \in C([0, T])$ such that $y_{n_k} \to y$ uniformly. In particular, $y(0) = y_0$. To show that y has the desired integral form, we write

$$\begin{aligned}\left|y(t) - y_0 - \int_0^t F(s, y(s))\ ds\right| &\leq |y(t) - y_{n_k}(t)| \\ &\quad + \left|y_{n_k}(t) - y_0 - \int_0^t F(s, y_{n_k}(s))\ ds\right| \\ &\quad + \int_0^t |F(s, y_{n_k}(s)) - F(s, y(s))|\ ds\end{aligned}$$

and show that each term can be made small. Fixing $\epsilon > 0$, the first term is made small simply by convergence of y_{n_k} to y. For the second term, we first note that the above estimates give

$$|s - t_k| + |y_n(s) - y_n(t_k)| \leq \frac{(1 + M)}{n} T.$$

Thus, for sufficiently large n, uniform continuity of F on R gives

$$|F(s, y_n(s)) - F(t_k, y_n(t_k))| < \frac{\epsilon}{T}$$

for every index k and every $s \in [t_k, t_{k+1}]$. Fix $t \in [0, T]$ and choose $0 \leq k \leq n - 1$ such that $t \in [t_k, t_{k+1}]$. Then by telescoping,

$$y_n(t) - y_0 = \sum_{j=0}^{k-1} (t_{j+1} - t_j) F(t_j, y_n(t_j)) + (t - t_k) F(t_k, y_n(t_k)).$$

Thus, splitting the integral $\int_0^t F(s, y_{n_k}(s))\ ds$ into the subintervals and subtracting, we obtain

$$\left|y_k(t) - y_0 - \int_0^t F(s, y_{n_k}(s))\ ds\right| \leq \frac{t}{T}\epsilon < \epsilon,$$

so that the second term can be made small.

For the last term, we simply use uniform continuity and that $(s, y_{n_k}(s)), (s, y(s)) \in R$ to make the difference in the integrand less than ϵ/T. Hence,

$$y(t) = y_0 + \int_0^t F(s, y(s))\ ds,$$

which is C^1 and solves the ODE, by the fundamental theorem of calculus. $\square$

Note that this is an improvement on the usual Picard existence theorem for ODEs, which requires F to be Lipschitz continuous and thus guarantees uniqueness (which is not included in the Peano theorem!).