Honors Analysis I

Recitation 11 Notes

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1. Concept Review

Last time we proved the Arzelà-Ascoli theorem and used it to establish Peano's existence theorem for ODEs. Today we briefly review the main theorem, work through several further examples, and present solutions to the quiz problems.

Let (X, d) be a compact metric space and let $C(X)$ denote the space of real-valued continuous functions on X with the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Recall that for $\mathcal{F} \subseteq C(X)$:

- $\mathcal{F}$ is *relatively compact* in $C(X)$ if $\overline{\mathcal{F}}$ is compact in $C(X)$ (equivalently, every sequence in $\mathcal{F}$ admits a uniformly convergent subsequence).
- $\mathcal{F}$ is *bounded* in $C(X)$ if there exists $M > 0$ such that $\|f\|_\infty \leq M < \infty$ for all $f \in \mathcal{F}$.
- $\mathcal{F}$ is *(uniformly) equicontinuous* on X if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ and $x, y \in X$ with $d(x, y) < \delta$.

Theorem 1.1 (Arzelà-Ascoli)**.** *Let X be a compact metric space and let $\mathcal{F} \subseteq C(X)$. Then $\mathcal{F}$ is relatively compact in $C(X)$ if and only if $\mathcal{F}$ is bounded and equicontinuous. In particular, $\mathcal{F}$ is compact in $C(X)$ if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.*

In practice, equicontinuity is most often verified via a uniform Lipschitz or Hölder estimate: if there exist constants $C > 0$ and $\alpha \in (0, 1]$ such that $|f(x) - f(y)| \leq Cd(x, y)^\alpha$ for all $f \in \mathcal{F}$ and $x, y \in X$, then $\mathcal{F}$ is equicontinuous with $\delta = (\epsilon/C)^{1/\alpha}$. On $X = [a, b]$, a uniform derivative bound $\sup_{f\in\mathcal{F}} \|f'\|_\infty \leq L < \infty$ yields the $\alpha = 1$ case via the Mean Value Theorem.

2. Examples

Example 2.1. *Let $X = C([0, 1])$ equipped with $\|\cdot\|_\infty$, and consider the subset*

$$\mathcal{F} = \left\{ f \in C^1([0, 1]) : |f(0)| \leq 1 \text{ and } \int_0^1 |f'(t)|^2 \, dt \leq 1 \right\}.$$

Is $\mathcal{F}$ relatively compact in X? Is $\mathcal{F}$ compact in X?

Solution:

We claim that $\mathcal{F}$ is relatively compact in X, but *not* compact in X. Note first that $\sup_{f\in\mathcal{F}} \|f'\|_\infty = \infty$ (a C^1 function can have a tall, narrow derivative spike while maintaining $\int_0^1 |f'(t)|^2 \, dt \leq 1$), so we cannot appeal to a uniform bound on $\|f'\|_\infty$. Instead, fix

$f \in \mathcal{F}$ and $0 \le s \le t \le 1$. By the Fundamental Theorem of Calculus and Cauchy-Schwarz,

$$|f(t) - f(s)| = \left| \int_s^t 1 \cdot f'(u)\, du \right| \le \left(\int_s^t 1^2\, du \right)^{1/2} \left(\int_s^t |f'(u)|^2\, du \right)^{1/2} \le |t-s|^{1/2}.$$

By symmetry, $|f(t) - f(s)| \le |t-s|^{1/2}$ for all $s, t \in [0,1]$ and $f \in \mathcal{F}$. Given $\epsilon > 0$, choosing $\delta = \epsilon^2$ gives $|f(t) - f(s)| < \epsilon$ whenever $|t-s| < \delta$, so $\mathcal{F}$ is equicontinuous. Moreover, setting $s = 0$ upgrades the single-point bound $|f(0)| \le 1$ to $\|f\|_\infty \le 2$, since $|f(t)| \le |f(0)| + |f(t) - f(0)| \le 1 + t^{1/2} \le 2$ for all $t \in [0,1]$. By Arzelà-Ascoli, $\mathcal{F}$ is relatively compact in $C([0,1])$.

To see that $\mathcal{F}$ is not compact, we show that $\mathcal{F}$ is not closed in $C([0,1])$. For each $n \in \mathbb{N}$, define

$$f_n(t) = \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{n^2}} - \sqrt{\frac{1}{4} + \frac{1}{n^2}}, \qquad t \in [0,1].$$

Each f_n lies in $C^1([0,1])$ with $f_n(0) = 0$, and for all $t \in [0,1]$,

$$|f_n'(t)| = \frac{|t - 1/2|}{\sqrt{(t-1/2)^2 + 1/n^2}} < 1 \implies \int_0^1 |f_n'(t)|^2\, dt < 1,$$

so $f_n \in \mathcal{F}$. Using $0 \le \sqrt{u^2 + a^2} - |u| = \frac{a^2}{\sqrt{u^2+a^2} + |u|} \le a$ $(a > 0)$, we see that $\|f_n - f\|_\infty \le 2/n \to 0$ as $n \to \infty$, where $f(t) = |t - 1/2| - 1/2$. Since f is not differentiable at $t = 1/2$, we have $f \notin C^1([0,1])$ and hence $f \notin \mathcal{F}$. Thus $\mathcal{F}$ is not closed in $C([0,1])$, hence not compact. □

Example 2.2. *Let* $X = C([0,\pi])$ *and* $E = \left\{ f_n(t) = \frac{\sin(nt)}{\sqrt{n}} \right\}_{n \in \mathbb{N}} \subseteq X$. *Is* E *a relatively compact subset of* X*?*

Solution:

In Example 2.1 of Recitation 10, we saw that $\{\sin(nt)\}_{n \in \mathbb{N}}$ is not equicontinuous on $[0,\pi]$. Here $f_n'(t) = \sqrt{n}\cos(nt)$, so $\|f_n'\|_\infty = \sqrt{n} \to \infty$. Nevertheless, E *is* relatively compact (and hence equicontinuous) in X, showing that uniform derivative bounds are sufficient for equicontinuity, not necessary.

First, $\|f_n - 0\|_\infty \le 1/\sqrt{n} \to 0$, so $f_n \to 0$ uniformly on $[0,\pi]$; hence $\overline{E} = E \cup \{0\}$ is sequentially compact in X, and Arzelà-Ascoli implies E is equicontinuous. Alternatively, combining the amplitude bound $|f_n(t) - f_n(s)| \le 2/\sqrt{n}$ with the Mean Value Theorem bound $|f_n(t) - f_n(s)| \le \sqrt{n}|t-s|$ via their geometric mean eliminates n directly:

$$|f_n(t) - f_n(s)| \le \left(\frac{2}{\sqrt{n}} \right)^{1/2} \left(\sqrt{n}|t-s| \right)^{1/2} = \sqrt{2}\, |t-s|^{1/2}$$

for all $t, s \in [0,\pi]$ and $n \in \mathbb{N}$. Since $\|f_n\|_\infty \le 1$, E is bounded and uniformly $1/2$-Hölder continuous, hence relatively compact by Arzelà-Ascoli. □

Example 2.3 (Fredholm Integral Operators)**.** *Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and define the integral operator $T : C([0,1]) \to C([0,1])$ by*

$$(Tf)(x) = \int_0^1 K(x,y)f(y)\,dy, \qquad x \in [0,1].$$

Let $\mathcal{B} = \{f \in C([0,1]) : \|f\|_\infty \leq 1\}$ be the closed unit ball in $C([0,1])$. Show that $T(\mathcal{B})$ is relatively compact in $C([0,1])$.

Solution:

Recall from Recitation 8 that $\mathcal{B}$ itself is not compact because $C([0,1])$ is infinite dimensional; however, integrating against the continuous kernel K smooths $\mathcal{B}$ into a bounded, equicontinuous family.

Fix $\epsilon > 0$. Since K is continuous on the compact square $[0,1]^2$, it is uniformly continuous: there exists $\delta > 0$ such that $|K(x_1,y_1) - K(x_2,y_2)| < \epsilon/2$ whenever $\|(x_1,y_1) - (x_2,y_2)\| < \delta$. Thus if $x_1, x_2 \in [0,1]$ satisfy $|x_1 - x_2| < \delta$, then $|K(x_1,y) - K(x_2,y)| < \epsilon/2$ for every $y \in [0,1]$, and for any $f \in C([0,1])$,

$$|(Tf)(x_1) - (Tf)(x_2)| \leq \int_0^1 |K(x_1,y) - K(x_2,y)||f(y)|\,dy \leq \frac{\epsilon}{2}\|f\|_\infty.$$

This shows both that $Tf \in C([0,1])$ for all $f \in C([0,1])$, and that $|(Tf)(x_1) - (Tf)(x_2)| \leq \epsilon/2 < \epsilon$ whenever $f \in \mathcal{B}$ and $|x_1 - x_2| < \delta$, so $T(\mathcal{B})$ is equicontinuous. Moreover, setting $M := \max_{(x,y)\in[0,1]^2} |K(x,y)| < \infty$, every $f \in \mathcal{B}$ and $x \in [0,1]$ satisfy

$$|(Tf)(x)| \leq \int_0^1 |K(x,y)||f(y)|\,dy \leq M\|f\|_\infty \leq M,$$

so $\|Tf\|_\infty \leq M$. By Arzelà-Ascoli, $T(\mathcal{B})$ is relatively compact in $C([0,1])$. □

Example 2.4. *(a) Let X be a compact metric space and suppose $\mathcal{F} \subseteq C(X)$ is equicontinuous and pointwise bounded (i.e. $\sup_{f\in\mathcal{F}} |f(x)| < \infty$ for every $x \in X$). Show that $\mathcal{F}$ is uniformly bounded, and hence relatively compact in $C(X)$.*

(b) Give an example of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions on $\mathbb{R}$ that is uniformly bounded and uniformly equicontinuous, yet has no uniformly convergent subsequence on $\mathbb{R}$.

Solution:

(a) Applying uniform equicontinuity of $\mathcal{F}$ with $\epsilon = 1$, choose $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $d(x,y) < \delta$ and $f \in \mathcal{F}$. By compactness of X, cover X by finitely many balls $B_\delta(x_1), \ldots, B_\delta(x_N)$. By pointwise boundedness, $M_i := \sup_{f\in\mathcal{F}} |f(x_i)| < \infty$, so $M := 1 + \max_{1\leq i\leq N} M_i < \infty$. For any $x \in X$ and $f \in \mathcal{F}$, choosing i with $x \in B_\delta(x_i)$ yields

$$|f(x)| \leq |f(x_i)| + |f(x) - f(x_i)| < M_i + 1 \leq M.$$

Hence $\|f\|_\infty \leq M$ for all $f \in \mathcal{F}$, so $\mathcal{F}$ is bounded and therefore relatively compact by Arzelà-Ascoli.

(b) Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{1}{1+(x-n)^2}$ for $n \in \mathbb{N}$. Clearly $\|f_n\|_\infty = 1$ for every $n \in \mathbb{N}$. Moreover, since $2|u| \le 1 + u^2$, we have

$$|f_n'(x)| = \frac{2|x-n|}{(1+(x-n)^2)^2} \le \frac{1}{1+(x-n)^2} \le 1 \qquad \text{for all } x \in \mathbb{R},$$

so $|f_n(x) - f_n(y)| \le |x-y|$ for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, making $\{f_n\}_n$ uniformly equicontinuous on $\mathbb{R}$. However, $\lim_{n\to\infty} f_n(x) = 0$ for each fixed $x \in \mathbb{R}$. If a subsequence $\{f_{n_k}\}_k$ converged uniformly on $\mathbb{R}$ to some f, pointwise convergence would force $f \equiv 0$, contradicting $\|f_{n_k} - 0\|_\infty \ge f_{n_k}(n_k) = 1$. Thus $\{f_n\}_n$ has no uniformly convergent subsequence on $\mathbb{R}$, showing that compactness of X in Arzelà-Ascoli cannot be dropped. □

3. Quiz Solutions

Question 1. Prove carefully that f is continuous at a if and only if $\operatorname{osc}_f(a) = 0$, directly from Definition 8.1 in the lecture notes:

$$\Omega_f(a,r) = \sup_{x,y \in B_r(a)} |f(x) - f(y)|, \qquad \operatorname{osc}_f(a) := \omega_f(a) = \inf_{r>0} \Omega_f(a,r).$$

Solution:

For every $r > 0$, taking $x = y = a \in B_r(a)$ gives $\Omega_f(a,r) \ge |f(a) - f(a)| = 0$, so 0 is a lower bound for $\{\Omega_f(a,r) : r > 0\}$ and hence $\operatorname{osc}_f(a) = \inf_{r>0} \Omega_f(a,r) \ge 0$ always.

First suppose f is continuous at a, and fix $\epsilon > 0$. Continuity at a applied to $\epsilon/3 > 0$ gives $\delta > 0$ such that $\|x - a\| < \delta \implies |f(x) - f(a)| < \epsilon/3$. For any $x, y \in B_\delta(a)$, the triangle inequality yields

$$|f(x) - f(y)| \le |f(x) - f(a)| + |f(a) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Taking the supremum over $x, y \in B_\delta(a)$ gives $\Omega_f(a,\delta) \le 2\epsilon/3 < \epsilon$, so $0 \le \operatorname{osc}_f(a) = \inf_{r>0} \Omega_f(a,r) \le \Omega_f(a,\delta) < \epsilon$. Since $\epsilon > 0$ was arbitrary, $\operatorname{osc}_f(a) = 0$.

Conversely, suppose $\operatorname{osc}_f(a) = \inf_{r>0} \Omega_f(a,r) = 0$, and fix $\epsilon > 0$. Since $\epsilon > 0$ is not a lower bound for $\{\Omega_f(a,r) : r > 0\}$, there exists $\delta > 0$ such that $\Omega_f(a,\delta) < \epsilon$. Then for every $x \in \mathbb{R}^d$ with $\|x - a\| < \delta$, both x and a lie in $B_\delta(a)$, so

$$|f(x) - f(a)| \le \sup_{u,v \in B_\delta(a)} |f(u) - f(v)| = \Omega_f(a,\delta) < \epsilon.$$

Therefore f is continuous at a. □

Question 9. In any metric space, show that every open set is an F_σ. (Hint: for U open, $U = \bigcup_n \{x : d(x, U^c) \ge 1/n\}$.)

Solution:

Let (X, d) be a metric space and let $U \subseteq X$ be open. Recall that a subset of X is an F_σ set if it can be written as a countable union of closed subsets of X. First note that if $U = X$, then X itself is closed in X, so $U = \bigcup_{n=1}^\infty X$ is trivially an F_σ set. Thus we may assume $U \subsetneq X$, so that $U^c := X \setminus U$ is nonempty and

$$d(x, U^c) := \inf_{y \in U^c} d(x, y)$$

is a well-defined nonnegative real number for every $x \in X$. Following the hint, define for each $n \in \mathbb{N}$ the set

$$F_n := \left\{ x \in X : d(x, U^c) \geq \frac{1}{n} \right\}.$$

We first show that $U = \bigcup_{n=1}^{\infty} F_n$. Suppose first that $x \in \bigcup_{n=1}^{\infty} F_n$, so $x \in F_n$ for some $n \in \mathbb{N}$. Then $d(x, U^c) \geq 1/n > 0$. If we had $x \in U^c$, then taking $y = x \in U^c$ in the infimum would give $0 \leq d(x, U^c) \leq d(x, x) = 0$, a contradiction; hence $x \notin U^c$, so $x \in U$. Conversely, suppose $x \in U$. Since U is open, there exists $r > 0$ such that $B_r(x) \subseteq U$. Every $y \in U^c$ then satisfies $y \notin B_r(x)$, meaning $d(x, y) \geq r$. Thus r is a lower bound for $\{d(x, y) : y \in U^c\}$, so

$$d(x, U^c) = \inf_{y \in U^c} d(x, y) \geq r > 0.$$

Choosing $n \in \mathbb{N}$ large enough that $1/n \leq r$, we obtain $d(x, U^c) \geq 1/n$, so $x \in F_n \subseteq \bigcup_{k=1}^{\infty} F_k$. Therefore $U = \bigcup_{n=1}^{\infty} F_n$.

To conclude the proof, we show each F_n is closed in X. Since U^c is nonempty, recall that the distance function $g : X \to \mathbb{R}$ defined by $g(x) = d(x, U^c)$ is continuous on X. Because $[1/n, \infty)$ is a closed subset of $\mathbb{R}$, its preimage $F_n = g^{-1}([1/n, \infty))$ is closed in X for every $n \in \mathbb{N}$. Hence $U = \bigcup_{n=1}^{\infty} F_n$ is a countable union of closed sets, i.e. an F_σ set. $\square$