Honors Analysis I

Recitation 2 Notes

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1. Quiz Solutions:

3. Prove that if $\{a_n\}_n$ is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Deduce that $\sum_{n=1}^{\infty} 1/n^s$ diverges if $0 < s < 1$.

Proof. Let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that for any $N \leq n < m$, we have by the Cauchy criterion for convergence

$$\left|\sum_{k=n}^{m} a_k\right| < \frac{\epsilon}{2}.$$

Now let $n \geq 2N$, so that $\lfloor n/2 \rfloor \geq \lfloor 2N/2 \rfloor = N$. Since $\{a_n\}_n$ is decreasing and positive, we obtain

$$\frac{n}{2}a_n \leq \left(n - \lfloor \frac{n}{2} \rfloor\right)a_n \leq \sum_{k=\lfloor n/2 \rfloor+1}^{n} a_k = \left|\sum_{k=\lfloor n/2 \rfloor+1}^{n} a_k\right| < \frac{\epsilon}{2}.$$

Thus, $|na_n| = na_n < \epsilon$ for all $n \geq 2N$, so that $\lim_{n\to\infty} na_n = 0$.

Notice that $\lim_{n\to\infty} n/n^s = \lim_{n\to\infty} n^{1-s} = \infty$ for $0 < s < 1$. Thus, since $1/n^s$ is decreasing and positive, it follows that $\sum_{n=1}^{\infty} 1/n^s$ diverges. □

4. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if for every $\epsilon > 0$, there exists a positive integer N such that

$$|a_{n_1} + \ldots + a_{n_k}| < \epsilon$$

for every finite subset $\{n_1, \ldots, n_k\}$ of positive integers $N \leq n_1 < n_2 < \ldots < n_k$.

Proof. First suppose $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then by the Cauchy criterion for convergence, given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for every $m > n \geq N$, we have

$$|a_n| + \ldots + |a_m| < \epsilon.$$

In particular, for an arbitrary finite subset $\{n_1, \ldots, n_k\}$ of integers $N \geq n_1 < \ldots < n_k$, positivity of each term above and the triangle inequality give

$$|a_{n_1} + \ldots + a_{n_k}| \leq |a_{n_1}| + \ldots + |a_{n_k}| \leq \sum_{j=n_1}^{n_k} |a_j| < \epsilon,$$

as desired.

For the converse, fix $\epsilon > 0$ and choose such an N as in the hypothesis such that

$$|a_{n_1} + \ldots + a_{n_k}| < \frac{\epsilon}{2}$$

for every finite subset $\{n_1, \ldots, n_k\}$. We fix $m > n \geq N$ and observe that

$$\begin{aligned}
\sum_{k=n}^{m} |a_k| &= \sum_{k,\ a_k>0} a_k + \sum_{k,\ a_k<0} (-a_k) \\
&= \left| \sum_{k,\ a_k>0} a_k \right| + \left| \sum_{k,\ a_k<0} a_k \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

so that $\sum_{n=1}^{\infty} a_n$ converges absolutely. $\square$

2. Homework Solutions

2. Assume that there exists an increasing function L from $[2, \infty)$ to $(0, \infty)$ which satisfies $L(x^n) = nL(x)$. Determine whether the following series converge or diverge.

(a) $\sum_{n=2}^{\infty} \frac{1}{nL(n)}$.

(b) $\sum_{n=2}^{\infty} \frac{1}{L(n)}$.

Proof. (a) We apply the Cauchy condensation test to show that the series diverges. Indeed, the regrouped series $2^k a_{2^k}$ where $a_n = 1/(nL(n))$ satisfies

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k L(2^k)} = \sum_{k=1}^{\infty} \frac{1}{kL(2)},$$

which is just the harmonic series, which we know to diverge. Therefore, since $a_n = 1/(nL(n))$ is monotone and $0 \leq a_{n+1} \leq a_n$ (since L is positive and increasing), we can apply the Cauchy condensation test to conclude that $\sum_{n=2}^{\infty} a_n$ diverges.

(b) The series $\sum_{n=2}^{\infty}$ also diverges, by comparison with the series in (a). The details are left as an exercise. $\square$

7. Let $a_n > 0$. Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof. The middle inequality is immediate. Let's show that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Let $\epsilon > 0$ be given and let $L = \limsup_{n\to\infty} a_{n+1}/a_n$. Then there is $N \in \mathbb{N}$ such that for all $n \geq N$ it follows that $a_{n+1}/a_n < L + \epsilon$. Now for $n > N$, we have

$$a_n = a_N \frac{a_{N+1}}{a_N} \cdots \frac{a_n}{a_{n-1}} \leq a_N (L+\epsilon)^{n-N} = \frac{a_N}{(L+\epsilon)^N} (L+\epsilon)^n.$$

Taking the n-th root and the limsup of this inequality, we obtain

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq L + \epsilon.$$

However, $\epsilon > 0$ is arbitrary, so we may take it to zero, proving the claim. The proof is identical for the lim inf case, and is left as an exercise. $\square$

10. A real sequence $\{a_n\}_n$ is said to be of bounded variation if

$$\sum_{k=1}^{\infty} |a_{k+1} - a_k| < \infty.$$

Show that if $\{a_k\}_k$ is of bounded variation, then $\{a_k\}_k$ converges. Is the converse true?

Proof. We show that $\{a_n\}_n$ is a Cauchy sequence. Fix $\epsilon > 0$. By the Cauchy criterion for series, there is $N \in \mathbb{N}$ such that for any $m > n \geq N$,

$$\sum_{k=n}^{m} |a_{k+1} - a_k| < \epsilon.$$

Observe that we can rewrite a_n as a telescoping sum

$$a_n = a_1 + \sum_{k=2}^{n} (a_k - a_{k-1}),$$

so that

$$|a_m - a_n| = \left| \sum_{k=n+1}^{m} (a_k - a_{k-1}) \right| \leq \sum_{k=n+1}^{m} |a_k - a_{k-1}| < \epsilon,$$

by the triangle inequality. Hence $\{a_n\}_n$ converges.

The converse is false. For instance, let $a_n = (-1)^n/n$. Then a_n converges yet

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{k+1} + \frac{1}{k} \right)$$

diverges. $\square$

12. Given a positive sequence $\{a_n\}_n$ such that $a_{n+1} \leq \left(\frac{n}{n+1}\right)^p a_n$ for some $p > 1$, show that $\sum_{n=1}^{\infty} a_n$ converges.

Proof. First, by induction, we have

$$a_n \leq a_1 \left(\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n-1}{n} \right)^p = \frac{a_1}{n^p}.$$

Thus, by comparison with the convergent series $\sum_{n=1}^{\infty} 1/n^p$, the series $\sum_{n=1}^{\infty} a_n$ is convergent. $\square$