Honors Analysis I

Recitation 3 Notes

Grayson Davis

1. Concept Review

Today we'll focus on the summation by parts formula and its consequences.

Theorem 1.1 (Summation by parts)**.** *Let $\{a_n\}_n$, $\{b_n\}_n$ be sequences and let $A_N = \sum_{n=1}^N a_n$. Then*

$$\sum_{n=1}^{N} a_n b_n = A_N b_N + \sum_{n=1}^{N-1} A_n(b_n - b_{n+1}).$$

More generally, for any $m > n$ and $A_0 \coloneqq 0$,

$$\sum_{k=n}^{m} a_k b_k = A_m b_m - A_{n-1} b_n + \sum_{k=n}^{m-1} A_k(b_k - b_{k+1}).$$

Note that $a_n = A_n - A_{n-1}$, so that the SBP formula becomes

$$\sum_{n=1}^{N} b_n(A_n - A_{n-1}) = A_N b_N - \sum_{n=1}^{N-1} A_n(b_{n+1} - b_n).$$

Now this looks a lot like our favourite formula from calculus, integration by parts:

$$\int b dA = Ab - \int A db,$$

so you can think of SBP as a discrete version of IBP.

Summation by parts is most useful when you can deconstruct a given series $\sum_{n=1}^\infty c_n$ into the sum of a product $\sum_{n=1}^\infty a_n b_n$ (i.e. $c_n = a_n b_n$) where a_n is oscillatory and b_n is monotone decreasing to zero. The importance of a_n being oscillatory is that the oscillations often provide sufficient cancellation so that the sequence of partial sums A_n remain uniformly bounded in n. For instance, $\sum_{n=1}^N \sin(n)$ is uniformly bounded (see Homework 3). This motivates the following theorem, called Dirichlet's test:

Theorem 1.2 (Dirichlet's test)**.** *Let $\{a_n\}_n$, $\{b_n\}_n$ be sequences such that the partial sums $\sum_{n=1}^N a_n$ are uniformly bounded in N and $b_n \searrow 0$. Then $\sum_{n=1}^\infty a_n b_n$ converges.*

Proof. Our goal is to show that we can simply take $N \to \infty$ in the SBP formula. Indeed, since

$$\{A_N\}_N = \left\{\sum_{n=1}^N a_n\right\}_N$$

is uniformly bounded, we have

$$|A_N b_N| \le \sup_{k\in\mathbb{N}} |A_k||b_N| = \sup_{k\in\mathbb{N}} |A_k| b_N \to 0$$

as $N \to \infty$, where we have used that $b_N \ge 0$ and $\lim_{N\to\infty} b_N = 0$. For the second term in the SBP formula, we use that $\{b_n\}_n$ is monotone decreasing, hence $b_n - b_{n+1} \ge 0$, to compute

$$\left|\sum_{n=1}^{N-1} A_n(b_n - b_{n+1})\right| \le \sup_{k\in\mathbb{N}} |A_k| \sum_{n=1}^{N-1}(b_n - b_{n+1}) = \sup_{k\in\mathbb{N}} |A_k|(b_1 - b_N).$$

Thus, taking $N \to \infty$ in the SBP formula, we obtain

$$\left|\sum_{n=1}^{\infty} a_n b_n\right| \le \sup_{k\in\mathbb{N}} |A_k| b_1 < \infty.$$

□

An immediate corollary of this is the alternating series test: if $\{b_n\}_n$ monotonically decreases to zero, then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Another particularly useful corollary of Dirichlet's test is Abel's test:

Theorem 1.3 (Abel's test)**.** *Suppose $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}_n$ is a bounded and monotone sequence. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.*

Proof. Since $\{b_n\}_n$ is bounded and monotone, the sequence converges to some limit b. Suppose without loss of generality that $b_n \searrow b$. Then

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} a_n(b + (b_n - b)) = \left(\sum_{n=1}^{N} a_n\right) b + \sum_{n=1}^{N} a_n(b_n - b).$$

Since $\sum_{n=1}^{\infty} a_n$ converges, the sequence of partial sums must be bounded. Moreover, since $\{b_n\}_n$ is monotone and $b_n \to b$, we have that $b_n - b \searrow 0$. Thus, we may apply the Dirichlet test to conclude that the second sum on the RHS converges. Taking the limit $N \to \infty$ concludes the proof. □

2. Quiz Solutions

1.

(a) Show that for every real number $x \ne 2\pi m$ ($m \in \mathbb{Z}$),

$$S_n(x) = \sum_{k=1}^{n} e^{ikx} = e^{i(n+1)x/2}\frac{\sin(nx/2)}{\sin(x/2)}, \quad \text{hence } |S_n(x)| \le 1/|\sin(x/2)|.$$

(b) Show that $\sum_{n=1}^{\infty} \sin(nx)/n$ converges for every $x \in \mathbb{R}$, and that $\sum_{n=1}^{\infty} \cos(nx)/n$ converges if and only if $x \notin 2\pi\mathbb{Z}$.

(c) Show that $\sum_{n=1}^{\infty} |\sin(n)|/n$ diverges. Conclude that $\sum_{n=1}^{\infty} \sin(n)/n$ converges conditionally. *Hint: Show first that* $|\sin(x)| \ge \sin^2(x) = (1 - \cos(2x))/2$ *then use (b).*

Proof. (a) Given $x \neq 2\pi m$ for any $m \in \mathbb{Z}$, we sum the geometric series to compute

$$S_n(x) = \sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{1-e^{inx}}{1-e^{ix}} = e^{ix}\frac{e^{inx/2}(e^{-inx/2}-e^{inx/2})}{e^{ix/2}(e^{-ix/2}-e^{ix/2})}$$

$$= e^{i(n+1)x/2}\frac{-2i\sin(nx/2)}{-2i\sin(x/2)}$$

$$= e^{i(n+1)x/2}\frac{\sin(nx/2)}{\sin(x/2)}.$$

Since $|\sin(nx/2)| \leq 1$ and $\left|e^{i(n+1)x/2}\right| \leq 1$, we conclude $|S_n(x)| \leq 1/|\sin(x/2)|$.

(b) If $x = 2\pi m$ for any $m \in \mathbb{Z}$ then $\sin(nx)/n = 0$, and hence the sum converges. Otherwise, by part (a) the sequence of partial sums of $\sin(nx)$ is bounded, and the sequence $\{1/n\}_n$ is monotonically decreasing to zero, so the Dirichlet test implies convergence of $\sum_{n=1}^{\infty} \sin(nx)/n$.

Likewise, by part (a) the sequence of partial sums of $\cos(nx)$ is bounded if $x \neq 2\pi m$ for any $m \in \mathbb{Z}$, the Dirichlet test gives convergence of $\sum_{n=1}^{\infty} \cos(nx)/n$. If instead $x = 2\pi m$, then $\cos(n2\pi m) = 1$, and the sum becomes the harmonic series, which diverges.

(c) First observe that $|\sin(x)| \in [0, 1]$, so that

$$|\sin(x)| \geq \sin^2(x) = (1-\cos(2x))/2,$$

where the last equality is a standard trigonometric identity. Thus, the partial sums satisfy

$$\sum_{n=1}^{N} \frac{|\sin(n)|}{n} \geq \sum_{n=1}^{N} \frac{1}{2}\left(\frac{1}{n} - \frac{\cos(2n)}{n}\right) = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{N}\frac{\cos(2n)}{n}.$$

Taking $N \to \infty$, the first term is the harmonic series, which diverges, and the second term is converges by part (b), so that the right-hand side of the inequality becomes infinite. Hence, $\sum_{n=1}^{\infty} \sin(n)/n$ is only conditionally convergent. ☐

4. Let $\{\epsilon_n\}_n$ be periodic, i.e. $\epsilon_{n+p} = \epsilon_n$ for some positive integer p, and let $\{b_n\}_n$ be monotone decreasing with $b_n \to 0$.

(a) Show that if $\sum_{n=1}^{p} \epsilon_n = 0$, then $\sum_{n=1}^{\infty} \epsilon_n b_n$ converges.

(b) Show that $1 + 1/2 - 2/3 + 1/4 + 1/5 - 2/6 + \ldots$ (pattern $1, 1, -2$) converges, while $1 + 1/2 - 1/3 + 1/4 + 1/5 - 1/6 + \ldots$ (pattern $1, 1, -1$) diverges.

Proof. (a) Since $\{\epsilon_n\}_n$ is periodic, given a positive integer N (taken to be sufficiently large), we have

$$\left|\sum_{n=1}^{N} \epsilon_n\right| = \left|\sum_{n=kp+1}^{N} \epsilon_n\right| \leq \sum_{n=kp+1}^{N} |\epsilon_n| \leq p \max_{j=1..p} |\epsilon_j|$$

for some nonnegative integer k. Thus, the sequence of partial sums of the ϵ_n is bounded, and by the monotonicity and convergence assumption on the b_n, we can apply the Dirichlet test to conclude that $\sum_{n=1}^{\infty} \epsilon_n b_n$ converges.

(b) The pattern $1, 1, -2$ sum converges by applying part (a) with $b_n = 1/n$.

For the pattern $1, 1, -1$ sum, we have the partial sum satisfies

$$1 + \frac{1}{2} - \frac{1}{3} + \ldots = \sum_{k=0}^{N} \left(\frac{1}{3k+1} + \underbrace{\frac{1}{3k+2} - \frac{1}{3k+3}}_{\geq 0} \right) \geq \sum_{k=0}^{N} \frac{1}{3k+1},$$

so that the sum diverges in the limit $N \to \infty$. $\square$