

Honors Analysis I

Recitation 5 Notes

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1. DEFINITION REVIEW

Today we will talk about topology and the topology of metric spaces.

Recall that a topology $\mathcal{T}$ on a set X is a collection of subsets of X satisfying

- $\emptyset, X \in \mathcal{T}$,
- If $\{U_\alpha\}_\alpha \subseteq \mathcal{T}$ is a collection of sets in $\mathcal{T}$, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$,
- If $U_1, \dots, U_n \in \mathcal{T}$ is a finite collection of sets in $\mathcal{T}$, then $\bigcap_{k=1}^n U_k \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets in X , and $(X, \mathcal{T})$ is called a topological space.

If (X, d) is a metric space, it is naturally equipped with the *metric topology* induced by the metric d . We define a set $U \subseteq X$ to be open if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$. A point x with this property is called an *interior point*, so open sets are equivalently characterized as sets in which every element is an interior point.

Another important kind of sets in a metric space are *closed sets*, defined as the complement of open sets. In metric spaces, a closed set $A \subseteq X$ is usually defined as a set that contains all its *limit points*, i.e., a point x such that $B_r(x) \setminus \{x\}$ has nonempty intersection with A .

The last type of set we will talk about today are compact sets. A set $K \subseteq X$ is compact if every open cover of K admits a finite subcover. In $\mathbb{R}^n$ with the norm topology, this definition is equivalent to K being closed and bounded, but beware that in general metric spaces this is not true.

2. EXAMPLES

Example 2.1. Define the discrete metric on a non-empty set X as

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Show that every set $A \subseteq X$ is both open and closed.

Solution:

Let $A \subseteq X$ be given and assume A is non-empty. Given $x \in X$, we observe that

$$B_{1/2}(x) = \{x\} \subseteq A,$$

so that A is open. In particular, every singleton $\{y\}$ is open, so that the union

$$X \setminus \{x\} = \bigcup_{y \neq x} \{y\}$$

is open. Hence $\{x\} = (X \setminus \{x\})^c$ is closed.

□

Remark 2.1. *The discrete topology on a set X is defined by letting every subset of X be open, i.e. $\mathcal{T} = \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X . The previous example shows that the discrete metric induces the discrete topology on X .*

Example 2.2. *Consider $(X, d) = (\ell^1, \|\cdot\|_1)$. Is the subset A of sequences $\{a_n\}_n$ such that $\sum_{n=1}^{\infty} |a_n| \geq 1$ open in X ? Closed in X ? Compact in X ?*

Solution:

We claim that A is closed but neither compact nor open. Indeed, A is the complement of the open unit ball in ℓ^1 , and is thus closed.

To see why A is not open, observe that

$$e_1 = (1, 0, \dots, 0, \dots) \in A.$$

However, if $r > 0$ is given and $0 < \epsilon < \min(r, 1)$, then

$$b = (1 - \epsilon, 0, \dots, 0, \dots)$$

satisfies $d(e_1, b) = \epsilon < r$ and $\|b\|_1 = 1 - \epsilon < 1$. Thus, $b \in B_r(e_1)$ but $b \notin A$, so A is not open.

To show that A is not compact, we need to find an open cover of A that does not admit a finite subcover. We consider the open cover

$$\mathcal{U} = \{B_1(a)\}_{a \in A}$$

of A . Notice that each $e_n \in A$, where e_n has a one in the n -th position and otherwise is 0. Observe also that $e_n \in A$ for every $n \in \mathbb{N}$, and that $d(e_n, e_m) = 2$ for every $n, m \in \mathbb{N}$. We claim that each ball in $\mathcal{U}$ contains at most one e_n , and hence any finite subcover would be missing some elements of A . Assume on the contrary, and choose $B_1(a) \in \mathcal{U}$ and e_n, e_m such that $e_n, e_m \in B_1(a)$. Then

$$2 = d(e_n, e_m) \leq d(e_n, a) + d(a, e_m) < 1 + 1 = 2,$$

a contradiction. Hence $\mathcal{U}$ does not admit any finite subcover of A , so that A is not compact.

□

Example 2.3. *Suppose (M, d) is a compact metric space and $f: M \rightarrow M$. Assume that*

$$d(f(x), f(y)) < d(x, y) \text{ for all } x, y \in X, x \neq y.$$

Show that f has a fixed point in X , i.e. there is $x^ \in X$ such that $f(x^*) = x^*$.*

Solution:

Define $g: X \rightarrow \mathbb{R}$ by $g(x) = d(x, f(x))$. Then

$$|g(x) - g(y)| \leq d(x, y) + d(f(x), f(y)) \leq 2d(x, y)$$

by the triangle inequality. Hence g is a continuous function on a compact set, and so attains its minimum $g(x^*)$ at some $x^* \in M$. We show x^* is a fixed point of f . Suppose instead that $f(x^*) \neq x^*$. Then

$$g(f(x^*)) = d(f(x^*), f(f(x^*))) < d(x^*, f(x^*)) = g(x^*),$$

a contradiction. □

3. QUIZ SOLUTIONS

5.

- (a) Let $2^{\mathbb{N}}$ denote the set of all sequences of 0s and 1s. Show that $d(a, b) = \sum_{n \geq 1} 2^{-n} |a_n - b_n|$ is a metric on $2^{\mathbb{N}}$.
- (b) The Hilbert cube H^{∞} is the set of all real sequences with $|x_n| \leq 1$ for all n . Show that $\sum_{n \geq 1} 2^{-n} |x_n - y_n|$ is a metric on H^{∞} .

Proof. (a) We first show that $d: 2^{\mathbb{N}} \times 2^{\mathbb{N}} \rightarrow [0, \infty)$ is well-defined. The sum is of non-negative numbers, so d indeed maps into $[0, \infty]$. To show that the sum is finite, let $a, b \in 2^{\mathbb{N}}$ be given, then

$$|a_n - b_n| = \begin{cases} 1 & a_n = 0, b_n = 1 \text{ or } a_n = 1, b_n = 0 \\ 0 & a_n = 1, b_n = 1 \text{ or } a_n = 0, b_n = 0 \end{cases}.$$

Thus, the sum $d(a, b)$ is at most the geometric series of 2^{-n} , hence converges by comparison.

The symmetry of the metric is immediate by symmetry of the absolute value. If $a = b$, then $a_n = b_n$ for all n and hence $d(a, b) = 0$. Conversely, if $d(a, b) = 0$, then

$$0 \leq 2^{-n} |a_n - b_n| \leq d(a, b) = 0,$$

so that $|a_n - b_n| = 0$, or equivalently $a = b$. Lastly, the triangle inequality holds by convergence of the infinite sum and the triangle inequality for the absolute value. More precisely, if $a, b, c \in 2^{\mathbb{N}}$ are given, then

$$\begin{aligned} d(a, b) &= \sum_{n \geq 1} 2^{-n} |a_n - b_n| \leq \sum_{n \geq 1} 2^{-n} (|a_n - c_n| + |b_n - c_n|) \\ &= \sum_{n \geq 1} 2^{-n} |a_n - c_n| + \sum_{n \geq 1} 2^{-n} |b_n - c_n| \\ &= d(a, c) + d(b, c). \end{aligned}$$

Thus, d is a metric on $2^{\mathbb{N}}$.

(b) The proof is more or less the same as in (a). The only difference is to show that d is well-defined on H^{∞} you simply use the triangle inequality to bound d by the geometric series. □

13. Show that $\|A\| = \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |a_{ij}|^2 \right)^{1/2}$ is a norm on the vector space $\mathbb{R}^{n \times m}$ of all $n \times m$ real matrices $A = (a_{ij})$.

Proof. Since the sum and maximum is over nonnegative numbers, we have that $\|A\|$ is well-defined.

We first show that $\|A\| = 0$ if and only if $A = 0$. If $A = 0$, then every element in the sum is 0, hence $\|A\| = 0$. Conversely, if $\|A\| = 0$, then assuming the maximum is attained at i , we obtain for every $1 \leq k \leq n$ and $1 \leq l \leq n$,

$$0 \leq |a_{kl}|^2 \leq \sum_{j=1}^m |a_{kj}|^2 \leq \sum_{j=1}^m |a_{ij}|^2 = 0.$$

Thus, $a_{kl} = 0$ for every k, l , so that $A = 0$.

To show absolute homogeneity of $\|\cdot\|$, let $\alpha \in \mathbb{R}$ be given and observe that

$$\begin{aligned} \|\alpha A\| &= \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |\alpha a_{ij}|^2 \right)^{1/2} = \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |\alpha|^2 |a_{ij}|^2 \right)^{1/2} \\ &= \max_{1 \leq i \leq n} \left(|\alpha|^2 \sum_{j=1}^m |a_{ij}|^2 \right)^{1/2} \\ &= |\alpha| \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |a_{ij}|^2 \right)^{1/2} \\ &= |\alpha| \|A\|. \end{aligned}$$

Lastly, to show subadditivity, let $A, B \in \mathbb{R}^{n \times m}$ be given and observe that

$$\begin{aligned} \|A + B\|^2 &= \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |a_{ij} + b_{ij}|^2 \right) \\ &\leq \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |a_{ij}|^2 + |b_{ij}|^2 + 2|a_{ij}||b_{ij}| \right) \\ &\leq \|A\|^2 + \|B\|^2 + 2 \max_{1 \leq i \leq n} \left(\sum_{j=1}^m |a_{ij}||b_{ij}| \right) \\ &\leq \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| \\ &= (\|A\| + \|B\|)^2 \end{aligned}$$

by the Cauchy-Schwarz inequality. Taking the square-root on both sides completes the proof. $\square$