

Honors Analysis I

Recitation 6 Notes

GRAYSON DAVIS

1. CONCEPT REVIEW

Let (X, d_X) , (Y, d_Y) be metric spaces and let $f: X \rightarrow Y$ be a mapping of X to Y . We call f continuous if any of the following equivalent statements hold:

- (1) (ϵ - δ continuity) Given $x \in X$, then for any $\epsilon > 0$ there is $\delta_x > 0$ such that if $y \in X$ is such that $d_X(x, y) < \delta_x$, then $d_Y(x, y) < \epsilon$.
- (2) (Sequential continuity) Given $x \in X$, if $\{x_n\}_n$ is a sequence in X converging to x , then $f(x_n) \rightarrow f(x)$ in Y .
- (3) (Topological continuity) For every open set V in Y , the set $f^{-1}(V)$ is open in X .

Let's do three examples that illustrate each definition's use.

Example 1.1. Let (X, d) be a metric space and let $A \subseteq X$ be non-empty. Define the distance function $d: X \rightarrow [0, \infty)$ by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Show that d is continuous.

Solution:

Fix $\epsilon > 0$ and suppose $x, y \in X$ are given so that $d(x, y) < \epsilon$. By the triangle inequality, we have for every $a \in A$ that

$$d(x, a) \leq d(x, y) + d(y, a),$$

and hence taking the infimum over all $a \in A$ gives

$$d(x, A) \leq d(x, y) + d(y, A).$$

Thus, by symmetry,

$$|d(x, A) - d(y, A)| \leq d(x, y) < \epsilon,$$

as desired.

□

Example 1.2. Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

Is f continuous at the origin?

Solution:

We claim that f is not continuous at the origin. It suffices to find a sequence $\{(x_n, y_n)\}_n \subseteq \mathbb{R}^2$ such that $(x_n, y_n) \rightarrow (0, 0)$ yet $f(x_n, y_n) \not\rightarrow (0, 0)$. Indeed, choosing $(x_n, y_n) = (1/n, 1/n)$, we see that $f(x_n, y_n) = 1/2$ for every $n \in \mathbb{N}$, so that f is not continuous at the origin.

□

Example 1.3. Define $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}.$$

Show that f is continuous on its domain.

Solution:

Given any open set $U \subseteq \mathbb{R}$, there are only a few possibilities for $f^{-1}(U)$, since the image of f is just $\{-1, 1\}$. Namely,

$$f^{-1}(U) = \begin{cases} \emptyset & -1, 1 \notin U \\ (-\infty, 0) & -1 \in U, 1 \notin U \\ (0, \infty) & -1 \notin U, 1 \in U \\ (-\infty, 0) \cup (0, \infty) & -1, 1 \in U \end{cases}.$$

In each of the above cases the pre-image is open, hence f is continuous.

□

2. MORE EXAMPLES

Example 2.1. For which $x \in \mathbb{R}$ are each of the following functions continuous:

(a) The Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

(b) The function $g(x) = xf(x)$, i.e.

$$g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

(c) Thomae's function (aka the Stars over Babylon function)

$$h(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms, } q > 0 \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Solution:

(a) By denseness of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$, given any $x \in \mathbb{R}$ we may choose a rational sequence $\{r_n\}_n$ and an irrational sequence $\{s_n\}_n$ such that $r_n, s_n \rightarrow x$ as $n \rightarrow \infty$. However, $f(r_n) = 1$ and $f(s_n) = 0$ for every $n \in \mathbb{N}$, so it is impossible for both sequences to converge to $f(x)$. Hence f is nowhere continuous.

(b) We claim that g is only continuous at $x = 0$. The same argument as in (a) shows that g is discontinuous at any $x \neq 0$. However, if $x = 0$, then

$$|g(y) - g(0)| = |y - 0| = |y|,$$

which is less than any $\epsilon > 0$ assuming $|y| < \epsilon$.

(c) We claim that h is continuous only at the irrational numbers. If x is rational, then the same argument as in (a) shows that h is discontinuous at x . Suppose instead x is irrational. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. We observe that the set

$$A = \left\{ \frac{p}{q} \in [x - 1, x + 1] : p \in \mathbb{Z}, 1 \leq q \leq N \right\}$$

is finite, and hence $d = \min_{r \in A} |x - r|$ is well-defined. Moreover, x is irrational, so that $x \notin A$ and $d > 0$. Choose $\delta = \min(d/2, 1/2)$. Then

$$B_\delta(x) \cap A = \emptyset \text{ and } B_\delta(x) \subseteq [x - 1, x + 1].$$

Now fix $y \in \mathbb{R}$ such that $|x - y| < \delta$. If y is irrational, then $|h(y) - h(x)| = |0 - 0| = 0$, and we are done. If instead $y = p/q$ is rational in lowest terms, then since $y \in B_\delta(x)$, we have $y \notin A$, and hence $q > N$. Thus,

$$|h(y) - h(x)| = \frac{1}{q} < \frac{1}{N} < \epsilon,$$

so that h is continuous at x .

□

Another interesting continuous function is the Cantor function. Recall from class that the Cantor set C is constructed by first removing the middle third of $[0, 1]$, then removing the middle thirds of the two remaining subintervals, and so on – so it is the intersection of all subintervals after successively removing the middle thirds of the subintervals at the step before. An equivalent definition is that

$$C = \left\{ x \in [0, 1] : x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} \text{ where } \alpha_n \in \{0, 2\} \right\},$$

i.e. C consists of all points of $[0, 1]$ whose ternary expansion has no ones.

Exercise 2.1. Show that these two definitions of C are in fact equivalent.

We define the Cantor function (aka the Devil's staircase) $c(x)$ as follows. If

$$x = \sum_{n=1}^{\infty} \frac{2}{3^n} \beta_n \in C, \beta_n \in \{0, 1\},$$

then define

$$c(x) := \sum_{n=1}^{\infty} \frac{\beta_n}{2^n}.$$

Otherwise, $[0, 1] \setminus C$ is the union of disjoint open intervals. Given any such removed interval (a, b) and any $x \in (a, b)$ define $c(x) = c(a) = c(b)$.

Exercise 2.2. *Show that $c(a) = c(b)$, so that c is well-defined.*

Thus, the Cantor function is defined piecewise by

$$c(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{\beta_n}{2^n} & x = \sum_{n=1}^{\infty} \frac{2}{3^n} \beta_n \in C \\ c(a) & x \in (a, b), \text{ where } (a, b) \text{ is a removed interval} \end{cases}.$$

Example 2.2. *Show that the Cantor function $c: [0, 1] \rightarrow [0, 1]$ is continuous.*

Solution:

The key observation is that at stage N of the Cantor set construction, each of the remaining 2^N intervals has length 3^{-N} . At stage N , we partition $[0, 1]$ into the 3^N subintervals

$$I_k = \left[\frac{k}{3^N}, \frac{k+1}{3^N} \right].$$

We show that $|c(x) - c(y)| \leq 2^{-N}$ for every $x, y \in I_k$. There are two cases. First, if I_k is in one of the remaining subintervals at stage N of the Cantor construction, then the ternary expansions of x and y agree up to the N -th element. Therefore,

$$|c(x) - c(y)| = \left| \sum_{n=N+1}^{\infty} \frac{\beta_n}{2^n} \right| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} = 2^{-N}.$$

If instead x, y are in one of the removed intervals, then since c is constant on these intervals,

$$|c(x) - c(y)| = 0 \leq 2^{-N}.$$

Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $2^{1-N} < \epsilon$. Let $\delta = 3^{-N}$ and suppose $|x - y| < \delta$. Then either x, y are in the same subinterval I_k , and we are done by the above argument. If instead x, y are in neighbouring subintervals, then they share a common endpoint $z \in I_k \cap I_{k+1}$. Then by the triangle inequality,

$$|c(x) - c(y)| \leq |c(x) - c(z)| + |c(z) - c(y)| \leq 2^{1-N} < \epsilon,$$

as desired. Therefore, the Cantor function c is continuous.

□

Remark 2.1. *In fact, the above proof shows that the Cantor function is uniformly continuous on $[0, 1]$. Despite being continuous and piecewise constant, the Cantor function is only differentiable almost everywhere – differentiability fails at points in the Cantor set itself, which is a measure zero set.*

3. QUIZ SOLUTIONS

18.

- (a) For any $r > 0$, show that the “closed ball” $\overline{B}_r(x) = \{y \in M : d(x, y) \leq r\}$ is a closed set.
- (b) Give an example of a metric space where $\overline{B}_r(x) \neq \overline{B_r(x)}$. *Hint: Discrete topology.*
- (c) Show that in any normed vector space $(V, \|\cdot\|)$, the two do coincide:

$$\{x : \|x\| \leq 1\} = \overline{\{x : \|x\| < 1\}}.$$

Proof. (a) We show that the complement is open. Let

$$y \in M \setminus \overline{B}_r(x) = \{y \in M : d(x, y) > r\}.$$

Then, setting $r' = (d(x, y) - r)/2 > 0$, we see that, for any $z \in B_{r'}(y)$,

$$d(x, z) > d(x, y) - d(z, y) > d(x, y) - r' = \frac{d(x, y) + r}{2} > r.$$

Hence $z \in M \setminus \overline{B}_r(x)$, so that the complement is open.

(b) We consider $(\mathbb{R}, d)$ where d is the discrete metric

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Then $\overline{B_1(0)} = \overline{\{0\}} = \{0\}$ yet $\overline{B}_1(0) = \mathbb{R}$.

(c) By (a), the closed unit ball is topologically closed, so we immediately obtain

$$\overline{B_1(0)} \subseteq \overline{B}_1(0).$$

Conversely, suppose $x \in \overline{B}_1(0)$. If $\|x\| < 1$ then $x \in \overline{B_1(0)}$ and we are done. Suppose instead $\|x\| = 1$ and define $x_n = (1 - 1/n)x$. Then

$$\|x_n\| = \left(1 - \frac{1}{n}\right)\|x\| = 1 - \frac{1}{n} < 1,$$

so that $x_n \in B_1(0)$ for every $n \in \mathbb{N}$. Moreover,

$$\|x_n - x\| = \frac{\|x\|}{n} \rightarrow 0$$

as $n \rightarrow \infty$, so that x is a limit point of $B_1(0)$ and thus $x \in \overline{B_1(0)}$. □

20.

- (a) If $x \neq y$ in M , show that there are disjoint open sets $U, V \subseteq M$ such that $x \in U$, $y \in V$. Show moreover that U, V can be chosen such that $\overline{U} \cap \overline{V} = \emptyset$.
- (b) If $F \subseteq M$ is closed and $x \notin F$, show that there are disjoint open sets $U, V \subseteq M$ such that $x \in U$ and $F \subseteq V$.

Proof. (a) Let $r = d(x, y)/3$. We claim that $B_r(x)$ and $B_r(y)$ are disjoint. If instead $z \in B_r(x) \cap B_r(y)$, then

$$d(x, y) \leq d(x, z) + d(y, z) < r + r < d(x, y),$$

a contradiction. Hence the balls are disjoint.

Moreover, if $z \in \overline{U} \cap \overline{V}$, then similarly

$$d(x, y) \leq d(x, z) + d(y, z) \leq r + r = 2d(x, y)/3 < d(x, y),$$

a contradiction. Hence $\overline{U} \cap \overline{V} = \emptyset$.

(b) Since F is closed, $M \setminus F$ is open, and hence there is $r > 0$ such that $B_r(x) \subseteq M \setminus F$. We claim that $F \subseteq M \setminus \overline{B}_{r/2}(x)$, which is open (as it is the complement of a closed set) and by definition is disjoint from $B_{r/2}(x)$. Indeed, $\overline{B}_{r/2}(x) \subseteq B_r(x) \subseteq M \setminus F$, so any $y \in F$ cannot be in $\overline{B}_{r/2}(x)$. $\square$