

Honors Analysis I

Recitation 7 Notes

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1. CONCEPT REVIEW

1.1. Sequences of continuous functions. Today we'll talk about sequences of continuous functions and power series. The lecture notes considers the space of continuous functions $C[a, b]$ on the compact interval $[a, b]$. We'll start off these notes in a bit more generality then look at specific examples.

Let (X, d) be a compact metric space. Recall that a function $f: X \rightarrow \mathbb{R}$ is continuous if given $x \in X$ and $\epsilon > 0$ there is some $\delta_x > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta_x$. Let $\{f_n\}_n \subseteq C(X)$ be a sequence of continuous functions $f_n: X \rightarrow \mathbb{R}$. We say that $f_n \rightarrow f$

- *pointwise on X* if $f_n(x) \rightarrow f(x)$ for every $x \in X$.
- *uniformly on X* if $\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$.

Equivalently, $f_n \rightarrow f$ uniformly if given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that **for every** $x \in X$ $|f_n(x) - f(x)| < \epsilon$ for every $n \geq N$. That is, the N we find depends only on ϵ and not x – it is *uniform* in x . One particularly important reason we care about uniform continuity is that it preserves continuity under taking limits:

Proposition 1.1. *Let $\{f_n\}_n$ is a sequence of continuous functions $f_n: X \rightarrow \mathbb{R}$ on a metric space X (not necessarily compact). If $f_n \rightarrow f$ uniformly, then f is continuous.*

Proof. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/3$ for $n \geq N$ and all $x \in X$. Given any $x, y \in X$, the triangle inequality gives

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

The first and last terms in this inequality are smaller than $\epsilon/3$ by the uniform convergence. For the second term, since f_n is continuous, there is $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon/3$ if $d(x, y) < \delta$, which concludes the proof. $\square$

The reason compact metric spaces are so nice to work with is because we have the extreme value theorem, so $\sup_{x \in X} |f(x)| = \max_{x \in X} |f(x)|$ when X is compact. For compact X , the space $C(X)$ is complete in the norm $\|f\|_\infty = \max_{x \in X} |f(x)|$ (as shown in the lecture notes).

Example 1.1. *Let $f_n(x) = x^n$ defined on $[0, 1]$. Show that $\{f_n\}_n$ converges pointwise to some function f but not uniformly.*

Solution:

If $0 \leq x < 1$, then $f_n(x) = x^n \rightarrow 0$ as $n \rightarrow \infty$, so that

$$f_n(x) \rightarrow f(x) := \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

pointwise in x . By the above proposition, the convergence cannot be uniform, since f is not continuous on $[0, 1]$. □

It is also important to note that the converse of the above proposition fails: there are continuous functions that are the pointwise but not uniform limit of a sequence of continuous functions.

Example 1.2. Define $f_n: [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = (nx)/(1+n^2x^2)$. Show that $\{f_n\}_n$ converges pointwise but not uniformly to a continuous function.

Solution:

Fixing $x \in [0, 1]$, we simply have $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$, so that $\{f_n\}_n$ converges pointwise to the zero function (which is continuous). However, the convergence is not uniform. Indeed, let $x_n = 1/n$, so that $f_n(x_n) = 1/2$, and hence $\sup_{x \in [0, 1]} |f_n(x)| \geq 1/2$. Thus, $\{f_n\}_n$ cannot converge to the zero function uniformly. □

1.2. Power series. Given that the uniform limits of continuous functions are continuous, we can look at sequences of partial sums of continuous functions and study the uniform convergence of those sums. The most simple continuous functions are polynomials, so it is natural to first examine *power series* and functions defined by a such series. A power series centered at $x_0 \in \mathbb{R}$ is defined by

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where $a_n \in \mathbb{R}$ (or $\mathbb{C}$) are the coefficients. Without loss of generality, we'll consider the $x_0 = 0$ case.

Proposition 1.2. *There exists $R \in [0, \infty]$ a radius of convergence such that the power series converges absolutely if $|x| < R$ and diverges if $|x| > R$. On the boundary $|x| = R$ the convergence must be studied individually for each point.*

Proof. We note that $\limsup_{n \rightarrow \infty} |a_n|^{1/n}$ exists in $[0, \infty]$ for every sequence of coefficients $\{a_n\}_n$. Define

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}},$$

which is uniquely defined since the limsup exists. If $|x| < R$, then choose $0 < r < R$ such that $|x| < r$. By the definition of limsup, there is $N \in \mathbb{N}$ such that $|a_n|^{1/n} < 1/r$ for every

$n \geq N$, and hence

$$\sum_{n=0}^{\infty} |a_n x^n| < \sum_{n=0}^{\infty} \left(\frac{|x|}{r} \right)^n < \infty$$

since $|x|/r < 1$.

A similar argument shows the series diverges if $|x| > r$. □

One important result on power series is Abel's theorem:

Theorem 1.1 (Abel). *If $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and $\sum_{n=0}^{\infty} a_n$ converges, then*

$$\lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n,$$

i.e. the power series is continuous from the left at $x = 1$.

Example 1.3. *Given that*

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

for $|x| < 1$, determine whether or not

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Solution:

By the alternating series test, the sequence of coefficients converges, so by Abel's theorem the claim is true. □

2. QUIZ SOLUTIONS

31. On $X = C[0, 1]$, define $d_1(f, g) = \int_0^1 |f - g|$. Show that d_1 is a metric on X . Which axiom requires continuity, and where?

Proof. First note that d_1 is well-defined from $X \times X \rightarrow [0, \infty)$. Indeed, the integrand is positive so that the metric is positive, and given $f, g \in X$, we have by continuity of $|f - g|$ (which is continuous since f, g are continuous)

$$d_1(f, g) \leq (1 - 0) \max_{x \in [0, 1]} |f(x) - g(x)| < \infty.$$

Symmetry is obtained by symmetry of the absolute value, and the triangle inequality is a simple consequence of the triangle inequality for the absolute value plus linearity of the integral.

If $f = g$, then the integrand is zero, hence $d_1(f, g) = 0$. For the converse, suppose

$$d_1(f, g) = \int_0^1 |f - g| = 0.$$

We use continuity to show that $|f - g| = 0$. If instead there is $x_0 \in [0, 1]$ such that

$$c = |f - g|(x_0) > 0,$$

then by continuity there is $\delta > 0$ such that $|f - g|(y) > c/2$ in $B_\delta(x_0)$. Hence

$$\int_0^1 |f - g| \geq \int_{x-\delta/2}^{x+\delta/2} |f - g| \geq \delta c/2 > 0,$$

a contradiction. □

21/33. Let $(X, d_X), (Y, d_Y)$ be metric spaces. We call a function $f: X \rightarrow Y$ Lipschitz if there is a constant $C > 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for every $x, y \in X$.

(a) Show that any Lipschitz function $f: X \rightarrow Y$ is uniformly continuous.

(b) Show that $L(f) = \int_a^b f$ is Lipschitz from $(C[a, b], \|\cdot\|_\infty)$ to $\mathbb{R}$ with constant $C = b - a$.

(c) Deduce that $f_n \rightarrow f$ uniformly on $[a, b]$ implies $\int_a^b f_n \rightarrow \int_a^b f$.

Proof. (a) Given $\epsilon > 0$ choose $\delta = \epsilon/C$. Then for any $x, y \in X$ with $d_X(x, y) < \delta$, we have

$$d_Y(f(x), f(y)) \leq C d_X(x, y) < \epsilon,$$

as desired.

(b) By the triangle inequality and linearity of the integral,

$$|L(f) - L(g)| \leq \int_a^b |f - g| \leq \|f - g\|_\infty \int_a^b 1 = (b - a) \|f - g\|_\infty,$$

so that L is Lipschitz continuous with constant $b - a$.

(c) Suppose that $f_n \rightarrow f$ uniformly on $[a, b]$. Then by (b),

$$\left| \int_a^b f_n - \int_a^b f \right| \leq (b - a) \|f_n - f\|_\infty \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\int_a^b f_n \rightarrow \int_a^b f$. □