

Honors Analysis I

Recitation 8 Notes

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1. CONCEPT REVIEW

Today we'll discuss compactness in normed linear spaces. Recall that in $\mathbb{R}^d$, a set is compact if and only if it is closed and bounded. Moreover, the norm we equip $\mathbb{R}^d$ with is irrelevant, as all norms are equivalent. In fact, since all norms on a finite dimensional space are equivalent, any finite dimensional normed linear space is topologically (and algebraically) equivalent to $\mathbb{R}^d$.

Exercise 1.1. *Construct such an isomorphism from a d -dimensional normed space $(V, \|\cdot\|)$ to $\mathbb{R}^d$.*

We will show that the closed unit ball in a normed linear space $(X, \|\cdot\|)$ is compact if and only if X is finite dimensional. A key ingredient is the following:

Lemma 1.1 (Riesz Lemma). *Let $(X, \|\cdot\|)$ be a normed linear space, $Y \subsetneq X$ a closed proper subspace, and $0 < \alpha < 1$. Then there is a vector $x_0 \in X$ with $\|x_0\| = 1$ such that $\inf_{y \in Y} \|x_0 - y\| \geq \alpha$.*

Proof. Let $x \in X \setminus Y$ and let $d = \inf_{y \in Y} \|x - y\|$. Since Y is closed, we have $d > 0$, and since $0 < \alpha < 1$, there is $y_0 \in Y$ such that

$$d \leq \|x - y_0\| \leq \frac{d}{\alpha}.$$

Now given $y \in Y$, we have

$$\left\| \frac{x - y_0}{\|x - y_0\|} - y \right\| = \frac{1}{\|x - y_0\|} \|x - (y_0 + \|x - y_0\|y)\|.$$

Since Y is a subspace, we know that $y_0 + \|x - y_0\|y \in Y$, and thus

$$\|x - (y_0 + \|x - y_0\|y)\| \geq d.$$

Combining this with $\|x - y_0\| \leq d/\alpha$, we obtain

$$\left\| \frac{x - y_0}{\|x - y_0\|} - y \right\| \geq \frac{\alpha}{d} d = \alpha,$$

as desired. □

We also will use that in a metric space, compactness is equivalent to sequential compactness (see the Week 7 notes).

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then X is finite dimensional if and only if the closed unit ball in X is compact.*

Proof. The first implication is a simple consequence of every finite-dimensional space being isomorphic to $\mathbb{R}^d$ and the Heine-Borel theorem.

Conversely, suppose the closed unit ball is compact in X but X is not finite dimensional. We construct via the Riesz lemma a sequence $\{x_n\}_n$ in the closed unit ball (in particular, of unit norm) such that $\|x_n - x_m\| > 1/2$ for every $n, m \in \mathbb{N}$ with $n \neq m$. Such a sequence therefore cannot have a convergent subsequence, because no subsequence is Cauchy. The sequence $\{x_n\}_n$ is constructed inductively as follows. Choose any $x_1 \in X$ with $\|x_1\| = 1$. Given $x_1, \dots, x_n \in X$ with unit norm, define

$$M = \text{span}\{x_1, \dots, x_n\}.$$

Then M is a finite-dimensional subspace of X , and is hence closed. Since X is not finite dimensional, we have $M \subsetneq X$, so by the Riesz lemma there is $x_{n+1} \in X$ with $\|x_{n+1}\| = 1$ such that $\inf_{y \in M} \|x_{n+1} - y\| > 1/2$, completing the inductive construction of our sequence. Thus, we have a contradiction to compactness, so that X must be finite dimensional. $\square$

Remark 1.1. *Equivalently, a normed space is finite dimensional if and only if any closed and bounded subset is compact. To see why, observe first that if every closed and bounded subset is compact then the closed unit ball is compact and the construction above can be applied to show the space must be finite dimensional. If instead the space is finite dimensional, then we can translate and multiplicatively scale the closed unit ball (translations and scaling preserve compactness) to contain any closed and bounded set, and the conclusion follows from the compactness of closed subsets of compact sets.*

Note that this is a much more general result than what we have seen before in the ℓ^p spaces: the sequence $\{e_n\}_n$ lives in the closed unit ball yet has no convergent subsequence.

Let's do another example with the Riesz lemma:

Example 1.1. *Let X be a normed linear space, and $X_0 \subseteq X$ a subspace. Suppose there is a constant $0 < c < 1$ such that*

$$\inf_{x \in X_0} \|x - y\| \leq c\|y\| \text{ for any } y \in X.$$

Prove that X_0 is dense in X .

Solution:

Suppose instead that X_0 is not dense in X , so that $\overline{X_0} \subsetneq X$ is a closed subspace of X (as an exercise, show that the closure of a subspace is also a subspace). Thus, the Riesz lemma gives $x_0 \in X \setminus \overline{X_0}$ with unit norm such that

$$c < \frac{1+c}{2} \leq \inf_{y \in \overline{X_0}} \|x_0 - y\| \leq \inf_{y' \in X_0} \|x_0 - y'\| \leq c\|x_0\| = c,$$

a contradiction. $\square$

Example 1.2. Let X be a normed linear space and $X_0 \subseteq X$ a finite dimensional subspace. Show that for every $x \in X$ there is $y_0 \in X_0$ such that

$$\|x - y_0\| = \inf_{y \in X_0} \|x - y\|.$$

Such a y_0 is called a best approximation to x .

Solution:

If $x \in X_0$ then may simply choose $y_0 = x$, so assume $x \in X \setminus X_0$ and let $d = \inf_{y \in X_0} \|x - y\|$. Choose $y_n \in X_0$ such that

$$d \leq \|x - y_n\| < d + \frac{1}{n}.$$

Notice that $d \leq \|x\|$, as $0 \in X_0$, so that

$$\|y_n\| \leq \|x - y_n\| + \|x\| \leq d + \frac{1}{n} + \|x\| \leq 2\|x\| + 1.$$

That is, $\{y_n\}_n$ is a bounded sequence in the ball of radius $2\|x\| + 1$ in X_0 , hence $y_{n_k} \rightarrow y_*$ for some $y_* \in X_0$ by compactness in finite dimensions. Then since $\|\cdot\|$ is continuous, we have

$$d \leq \|x - y_*\| = \lim_{k \rightarrow \infty} \|x - y_{n_k}\| \leq \lim_{k \rightarrow \infty} \left(d + \frac{1}{n_k}\right) = d,$$

as desired. □

2. QUIZ SOLUTIONS

47. Define a pseudometric on $C(\mathbb{R})$ by $d_n(f, g) = \max_{|x| \leq n} |f(x) - g(x)|$, and a metric on $C(\mathbb{R})$ by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(f, g)}{1 + d_n(f, g)}.$$

Show that $C(\mathbb{R})$ is separable for the metric d .

Proof. We claim that $\mathbb{Q}[x]$, the space of polynomials in x with rational coefficients, is dense in $C(\mathbb{R})$ in the metric d . First note that $\mathbb{Q}[x] = \bigcup_{n \geq 0} \mathbb{Q}_n[x]$, where $\mathbb{Q}_n[x]$ is the space of rational coefficient polynomials of degree at most n . Thus, $\mathbb{Q}[x]$ is a countable union of countable sets, so is itself countable.

To see why $\mathbb{Q}[x]$ is dense in $C(\mathbb{R})$, let $f \in C(\mathbb{R})$ be given and fix $\epsilon > 0$. We find a polynomial $q \in \mathbb{Q}[x]$ such that $d(f, q) < \epsilon$. The idea is to use the Weierstrass approximation theorem on compact sets and rational approximation of real numbers. First, observe that

$$\frac{d_n(f, g)}{1 + d_n(f, g)} \leq 1$$

for every $f, g \in C(\mathbb{R})$. Thus, we may choose $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \frac{d_n(f, g)}{1 + d_n(f, g)} \leq \sum_{n=N+1}^{\infty} 2^{-n} < \frac{\epsilon}{2}.$$

Now, on the compact set $[-N, N]$, the Weierstrass approximation theorem gives a polynomial $p \in \mathbb{R}[x]$ such that $d_N(f, p) < \epsilon/4$. Write

$$p = a_0 + \dots + a_k x^k$$

for some coefficients $a_0, \dots, a_k \in \mathbb{R}$ and choose rationals $b_0, \dots, b_k \in \mathbb{Q}$ such that

$$|a_j - b_j| < \frac{\epsilon}{4(k+1)N^j}$$

for every $0 \leq j \leq k$. Then $d_N(p, q) < \epsilon/4$, and hence by the triangle inequality $d_N(f, q) < \epsilon/2$, where $q = b_0 + \dots + b_k x^k$. Thus, by monotonicity in n of d_n ,

$$\begin{aligned} d(f, q) &= \sum_{n=1}^N 2^{-n} \frac{d_n(f, q)}{1 + d_n(f, q)} + \sum_{n=N+1}^{\infty} 2^{-n} \frac{d_n(f, q)}{1 + d_n(f, q)} \\ &< \sum_{n=1}^N 2^{-n} d_N(f, q) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} \sum_{n=1}^N 2^{-n} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Thus, $\mathbb{Q}[x]$ is a countable dense subset of $C(\mathbb{R})$. □

48(a). Let $\mathcal{P}_n$ be the polynomials of degree at most n , as a subset of $C[a, b]$.

- (1) Show that $\mathcal{P}_n$ is a subspace of $C[a, b]$. (You may use the convention that the zero polynomial has degree $-\infty$).
- (2) Show that $\mathcal{P}_n$ has dimension $n + 1$ in $C[a, b]$.
- (3) Show that $\mathcal{P}_n$ is closed in $(C[a, b], \|\cdot\|_{\infty})$.

Proof. (a) We must verify that $\mathcal{P}_n$ has the zero element and is closed under addition and scalar multiplication.

Since the degree of the zero polynomial has degree $-\infty$, we have that $0 \in \mathcal{P}_n$. Let $p, q \in \mathcal{P}_n$ and $\alpha \in \mathbb{R}$ be given. Write

$$p = a_0 + \dots + a_n x^n, \quad q = b_0 + \dots + b_n x^n,$$

for some coefficients $a_i, b_i \in \mathbb{R}$, so that

$$p + q = (a_0 + b_0) + \dots + (a_n + b_n)x^n \in \mathcal{P}_n.$$

Likewise,

$$\alpha p = (\alpha a_0) + \dots + (\alpha a_n)x^n \in \mathcal{P}_n,$$

so that $\mathcal{P}_n$ is a subspace of $C[a, b]$.

(b) We claim that $\mathcal{B} = \{1, x, \dots, x^n\}$ is a basis of $\mathcal{P}_n$, so that $\dim(\mathcal{P}_n) = n + 1$. Since any polynomial can be written as a linear combination of the elements of $\mathcal{B}$, we have that $\mathcal{B}$ is a spanning set. To show linear independence, suppose that

$$p(x) = a_0 + \dots + a_n x^n = 0.$$

Then $p^{(n)}(x) = n!a_n = 0$, so that $a_n = 0$. But then $p^{(n-1)}(x) = (n-1)!a_{n-1} = 0$, hence $a_{n-1} = 0$. Continuing downwards, we see that $a_j = 0$ for all $0 \leq j \leq n$, thus $\mathcal{B}$ is linearly independent.

(c) To see why $\mathcal{P}_n$ is closed in $C[a, b]$, let $\{p_n\}_n \subseteq \mathcal{P}_n$ converge to some $f \in C[a, b]$. Since $\{p_n\}_n$ is convergent, it is Cauchy in $(\mathcal{P}_n, \|\cdot\|_\infty)$. But $\mathcal{P}_n$ is finite-dimensional, so in particular is complete in any norm on the space, so that there exists $p \in \mathcal{P}_n$ a limit of $\{p_n\}$. By uniqueness of limits, we have $f = p$. $\square$