Honors Analysis I

Recitation 9 Notes

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1. Concept Review

Today we'll discuss the Baire category theorem. Let's first review some basic definitions. Let (X, d) be a metric space and $A \subseteq X$. Then

- A is *dense* in X if $\overline{A} = X$.
- A is *nowhere dense* in X if $\overline{A}$ has empty interior, i.e. $(\overline{A})^\circ = \varnothing$. Equivalently, A is nowhere dense if $X \backslash \overline{A}$ is dense, or $(A^c)^\circ$ is dense, i.e. A^c contains an open dense set.
- A is *meagre* (or of the *first category*) if it is a countable union of nowhere dense sets.
- A is *non-meagre* (or of the *second category*) if it is not meagre.

Theorem 1.1 (Baire Category Theorem)**.** *Any nonempty complete metric space is of the second category in itself. That is, if $\{O_n\}_n$ is a countable collection of dense open sets in X, then $\bigcap_{n\in\mathbb{N}} O_n$ is dense in X.*

A simple consequence is that if $X = \bigcup_{n\in\mathbb{N}} F_n$ for some closed subsets $F_n \subseteq X$, then some F_n has nonempty interior.

Let's do some examples:

Example 1.1.

(a) Show that $\ell^1 \subseteq \ell^2$.

(b) Show that ℓ^1 is meagre in ℓ^2.

Solution:

(a) Let $\{x_n\}_n \in \ell^1$. Then $\{x_n\}_n$ converges to zero and is thus bounded, so that

$$\sum_{n=1}^{\infty} |x_n|^2 \leq \max_{n\in\mathbb{N}} |x_n| \sum_{n=1}^{\infty} |x_n| < \infty.$$

(b) Define

$$F_k := \left\{ x \in \ell^2 : \sum_{n=1}^{\infty} |x_n| \leq k \right\}$$

and observe that $\ell^1 = \bigcup_{k=1}^{\infty} F_k$. We show that each F_k is closed and has empty interior. First, let $\{x^{(m)}\}_m \subseteq F_k$ converge to $x \in \ell^2$. Observe that

$$\left|x_n^{(m)} - x_n\right|^2 \leq \left\|x^{(m)} - x\right\|_2^2,$$

so that $x_n^{(m)} \to x_n$ as $m \to \infty$ as a sequence of real numbers. Thus,

$$\sum_{n=1}^{\infty} |x_n| = \lim_{N\to\infty} \left(\lim_{m\to\infty} \sum_{n=1}^{N} \left|x_n^{(m)}\right| \right) \leq k,$$

so that $x \in F_k$. We show that F_k has is no where dense by contradiction. Assume instead $x \in F_k$ is an interior point, so that there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq F_k$. Let

$$z = \{z_n\}_n \coloneqq \{1/n\}_n \in \ell^2 \backslash \ell^1.$$

Choose an $N \in \mathbb{N}$ sufficiently large so that $\|z\|_2/N < \epsilon$. Then

$$\left\| (x + \frac{z}{N}) - x \right\|_2 < \epsilon,$$

hence $x + z/N \in B_\epsilon(x) \subseteq F_k$. However, $x + z/N \notin \ell^1$, as if it were, then

$$z = N((x + z/N) - x) \in \ell^1,$$

a contradiction. Hence $x + z/N \notin F_k$, so that F_k has no interior points.

□

Exercise 1.1. *Show that $C^1[0,1]$ is a meagre subset of $C[0,1]$.*

Example 1.2. *Let X be a vector space with a countably infinite (Hamel) basis (i.e. every element of X can be written as a finite linear combination of elements of a countable linearly independent set). Show that there is no norm on X that makes X a complete normed space.*

Solution:

Suppose instead $\|\cdot\|$ is a norm on X such that $(X, \|\cdot\|)$ is a complete normed linear space. Let $\{e_n\}_n$ be a basis of X and set $X_k = \text{span}\{e_1, \ldots, e_k\}$. Then $X = \bigcup_{k\in\mathbb{N}} X_k$. Note that X_k is closed for each $k \in \mathbb{N}$, since it is finite dimensional. By the BCT, there is some $m \in \mathbb{N}$ such that X_m has nonempty interior. In particular, there is $x_0 \in X_m$ and $r > 0$ such that $B_r(x_0) \subseteq X_m$. But X_m is a subspace of X, so we can translate the ball by $-x_0$ to obtain $B_r(0) \subseteq X_m$, hence (after rescaling) $X_m = X$, a contradiction to the infinite dimensionality of X

□

Another important consequence of the BCT is the uniform boundedness principle, saying that a pointwise bounded family of continuous functions in a complete metric space is in fact uniformly bounded on some nonempty open set:

Theorem 1.2 (Uniform Boundedness Principle)**.** *Let $\mathcal{F} \subseteq C(X)$ be a family of continuous functions on a nonempty complete metric space X, and suppose that for every $x \in X$ there is $M_x > 0$ such that*

$$\sup_{f \in \mathcal{F}} |f(x)| \leq M_x.$$

Then there is a nonempty open set $O \subseteq X$ and a constant $M > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{x \in O} |f(x)| \leq M.$$

Example 1.3 (Banach-Steinhaus Theorem)**.** *Let X be a complete normed linear space (i.e. a Banach space) and Y a normed linear space. Suppose $\{T_\alpha\}_\alpha$ is a family of bounded linear maps $X \to Y$ such that $\sup_\alpha \|T_\alpha x\|_Y < \infty$ for every $x \in X$. Prove that $\sup_\alpha \|T_\alpha\| < \infty$, where $\|T_\alpha\| = \sup_{x \in X, \|x\|_X = 1} \|T_\alpha x\|_Y$ is the operator norm of T_α.*

Solution:

For each α, define $f_\alpha \colon X \to \mathbb{R}$ by $f_\alpha(x) = \|T_\alpha x\|_Y$. Since T and $\|\cdot\|_Y$ are continuous, so is f_α. Moreover, by assumption

$$\sup_\alpha |f_\alpha(x)| = \sup_\alpha \|T_\alpha x\|_Y < \infty,$$

so that by the UBP we obtain an open set $O \subseteq X$ such that

$$\sup_\alpha \sup_{x \in O} \|T_\alpha x\|_Y = \sup_\alpha \sup_{x \in O} |f_\alpha(x)| \leq M$$

for some $M > 0$. It remains to find a uniform bound on the T_α over all $x \in X$ with $\|x\|_X = 1$. Since O is open, let $x_0 \in O$ and $r > 0$ be such that $B_r(x_0) \subseteq O$. We first get a bound on T_α over a ball near the origin, then rescale that ball to get a uniform bound on the operator norms of the T_α. If $x \in B_r(0)$, then $x + x_0 \in B_r(x_0)$, so that

$$\|T_\alpha x\|_Y \leq \|T_\alpha(x + x_0)\|_Y + \|T_\alpha x_0\|_Y \leq 2M.$$

Thus, $\sup_\alpha \sup_{x \in B_r(0)} \|T_\alpha x\|_Y \leq 2M$. Now given $x \in X$ with $\|x\|_X = 1$, we have, by linearity,

$$\|T_\alpha x\| = \frac{2}{r} \left\| T_\alpha \left(\frac{r}{2} x \right) \right\| \leq \frac{4M}{r},$$

which proves the theorem.

□

Example 1.4. *Let X, Y be Banach spaces and $\{T_n\}_n$ a sequence of bounded linear operators $T_n \colon X \to Y$ such that $\lim_{n \to \infty} T_n(x)$ exists in Y for every $x \in X$. Show that*

$$Tx \coloneqq \lim_{n \to \infty} T_n x$$

is a bounded linear operator $X \to Y$.

Solution:

To show linearity of T, let $x, y \in X$, $\alpha \in \mathbb{R}$ be given, and observe that

$$T(x + \alpha y) = \lim_{n \to \infty} T_n(x + \alpha y) = \lim_{n \to \infty} (T_n x + \alpha T_n y) = Tx + \alpha T y.$$

To show that T is bounded, we appeal to Banach-Steinhaus. First, observe that since $\{T_n x\}_n$ is a convergent sequence in Y, it is bounded. Thus, then Banach-Steinhaus gives that $\sup_{n\in\mathbb{N}} \|T_n\| \leq M < \infty$ for some $M > 0$. In particular, for every $x \in X$ with $\|x\| = 1$, we have $\|T_n x\|_Y \leq M$. By continuity of the norm, we can take $n \to \infty$ to conclude $\|Tx\| \leq M$ for every $x \in X$ with $\|x\| = 1$. Therefore, $T\colon X \to Y$ is a bounded linear operator.

□

We should also note that the Baire category theorem also holds for a specific class of topological spaces. Let X be a topological space. We call X locally compact if every $x \in X$ has a compact neighbourhood, i.e. there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U \subseteq K$. We call X Hausdorff if given distinct $x, y \in X$ there are disjoint open sets $U_x, U_y \subseteq X$ such that $x \in U_x$ and $y \in U_y$.

Theorem 1.3 (Baire Category Theorem)**.** *Any nonempty locally compact Hausdorff space is of the second category.*

The proof is similar to the metric space case, and is a good exercise to make sure you understand the proof.

2. Quiz Solutions

2. Prove that c_0, the space of sequences converging to 0, is complete, by showing that c_0 is closed in $(\ell^\infty, \|\cdot\|_\infty)$.

Proof. Since any closed subset of a complete metric space is complete, it suffices to show that c_0 is closed. Let $\{x^{(k)}\}_k \subseteq c_0$ and suppose $x^{(k)} \to x \in \ell^\infty$. We show that $x \in c_0$. Let $\epsilon > 0$ be given and choose $K \in \mathbb{N}$ such that $\left\|x^{(k)} - x\right\|_\infty < \epsilon/2$. Moreover, since $x^{(K)} \in c_0$, we may choose $N \in \mathbb{N}$ such that $\left|x_n^{(K)}\right| < \epsilon/2$ for every $n \geq N$. Thus,

$$|x_n| \leq \left|x_n - x_n^{(K)}\right| + \left|x_n^{(K)}\right| < \epsilon,$$

so that $x \in c_0$. □

20. Give an example of a closed, bounded subset of $(\ell^\infty, \|\cdot\|_\infty)$ that is not totally bounded, and prove your claim.

Proof. Let $A = \{e_n\}_n$ where the k-th element of e_n is zero if $k \neq n$ and one if $k = n$. We claim that A is closed and bounded but not totally bounded.

The key observation is that $\|e_n - e_m\|_\infty = 1$ if $n \neq m$, which follows from

$$(e_n - e_m)(k) = \begin{cases} 1 & k = n \\ -1 & k = m \\ 0 & k \neq n, m \end{cases}.$$

Thus, the distance between any two elements in A is bounded, so that A is bounded. Moreover, if $\{x^{(k)}\}_k \subseteq A$ converges to some $x \in \ell^\infty$, then since $\{x^{(k)}\}_k$ is Cauchy, there exists $K \in \mathbb{N}$ such that

$$\left\|x^{(k)} - x^{(j)}\right\|_\infty < \frac{1}{2} \text{ for all } k, j \geq K.$$

But distinct elements of A have distance 1, so $x^{(k)} = x^{(K)}$ for every $k \geq K$. Hence $x^{(k)} \to x = x^{(K)} \in A$, so that A is closed.

To see why A is not totally bounded, observe that any ball of radius $1/2$ contains at most one element of A, hence A cannot be covered by finitely many balls of radius $1/2$. Indeed, if $e_n, e_m \in B_{1/2}(x)$ for some point $x \in \ell^\infty$ and $n \neq m$, then

$$1 = \|e_n - e_m\|_\infty \leq \|e_n - x\|_\infty + \|x - e_m\|_\infty < 1,$$

a contradiction. $\square$