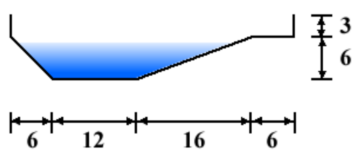
Most of these are seminar questions from Math 150 at Simon Fraser University (SFU). This document will be periodically updated over the Fall 2025 term.

1. Functions Review

1. (SFU) Consider the function $f(x) = \frac{|x^3+3x|}{x}$.
   (a) Find the domain of f.
   (b) Simplify the expression $\frac{|x^3+3x|}{x}$ for $x < 0$.
   (c) Let $g(x)$ be the result from part (b). What is the domain of g? Are f and g the same functions?
2. (SFU) A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure below.
   (a) Express the volume of the water in the pool as a function of height h of the water above the deepest point *Hint: the volume will be a piecewise-defined function.*
   (b) Find the domain and range of the function found in part (a).



3. (SFU) Let L be the line through the points $(0, -2)$ and $(2, 4)$ and f be the function given by $f(x) = x^2 - 2x + 2$.
   (a) Find the equation of the line L. Determine whether the points $(-1, -3)$ and $(\frac{7}{3}, 5)$ lie on L.
   (b) Determine whether the points $(3, 2)$ and $(\sqrt{3} + 1, 4)$ are on the graph of f.
   (c) Sketch the graph of L and f on the same coordinate axes.
   (d) Find the intersection points of L and the graph of $y = f(x)$.
4. (SFU) Let $A, B \subseteq \mathbb{R}$ be subsets of the real numbers. Suppose f has domain A and g has domain B.
   (a) What is the domain of $f + g$?
   (b) What is the domain of f, g?
   (c) What is the domain of f/g?

5. Let f be a function on some subset A of the real numbers $\mathbb{R}$ ($f\colon A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$). Define the pre-image of a set $B \subseteq \mathbb{R}$ under f as

$$f^{-1}(B) = \{x \in A : f(x) \in B\}.$$

Let $A, B \subseteq \mathbb{R}$. Prove the following claims:

(a) $f(A \cup B) = f(A) \cup f(B)$.

(b) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(e) Provide a counterexample to the claim $f(A \cap B) = f(A) \cap f(B)$.

2. Inverse Functions and Logarithms

1. (SFU) Let $g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

   (a) Find the domain of g.

   (b) Show that $g(x)$ is an odd function (i.e. $g(-x) = -g(x)$).

   (c) Why is $g(x)$ a one-to-one (injective) function?

   (d) Find the inverse of $g(x)$.

2. (SFU) Solve $\ln(x^2 - 1) = 3$ for x.

3. Define $f(x) = \frac{\ln\left(x^2\right)}{x^4 - 1}$.

   (a) Find the domain of f.

   (b) Determine, in interval notation, the set of all x such that $f(x) \geq 0$.

4. (SFU) Suppose f is a one-to-one (injective) function whose *inverse* is given by the formula
$$f^{-1}(y) = y^5 + 3y^3 + 2y + 1.$$

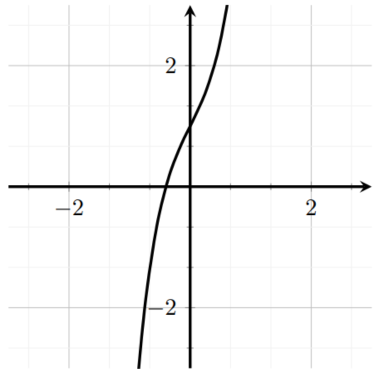
   (a) Compute $f^{-1}(1)$ and $f(1)$.

   (b) Compute the value of x_0 such that $f(x_0) = 1$.

   (c) Compute the value of y_0 such that $f^{-1}(y_0) = 1$.

   (d) Given the graph of f^{-1} below, graph f.

5. Let $f\colon A \to B$ be a function. Suppose f has an inverse. Prove that the inverse is unique *Hint: assume g, h are both inverses of f and show that $g = h$.*



3. Limits

1. (SFU) Sketch the graph of an example function that satisfies the following conditions:

$$\lim_{x\to 0} f(x) = 1, \quad \lim_{x\to 3^-} f(x) = -2, \quad \lim_{x\to 3^+} f(x) = 2, \quad f(0) = -1, \quad f(3) = 1.$$

2. (SFU) Is there a number b such that

$$\lim_{x\to 2} \frac{bx^2 - 10x + 10 + b}{x^2 - 2x - 2}$$

exists? If so, find b and evaluate the limit.

3. (SFU) Suppose that

$$x^4 < f(x) < x^2$$

if $|x| < 1$ and

$$x^2 < f(x) < x^4$$

if $|x| > 1$. Do the following limits exist? If so, find their value.

   (a) $\lim_{x\to -1} f(x)$.

   (b) $\lim_{x\to 0} f(x)$.

   (c) $\lim_{x\to 1} f(x)$.

4. (SFU) Evaluate the following limits:

   (a) $\lim_{x\to 10} \frac{x^2-100}{x-9}$.

   (b) $\lim_{x\to 10} f(x)$, where $f(x) = x^2$ for all $x \neq 10$ and $f(10) = 99$.

   (c) $\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8}$.

   (d) $\lim_{x\to 8} \frac{(x-8)(x-2)}{|x-8|}$.

5. The rigorous definition of limits is the following. Let f be a function defined in an interval containing a point $x = a$ (but possibly not defined at $x = a$). We say that

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$, there is a number $\delta_\epsilon > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta_\epsilon.$$

Essentially, f is arbitrarily close to L whenever x is close enough to a.

Use this definition to show that $\lim_{x \to 0} x^2 = 0$.

4. Continuity

1. (SFU) Define the function f by

$$f(x) = \begin{cases} 2x + 1 & x \leq 0 \\ bx^3 - 1 & 0 < x \leq 1 \\ x^2 + 2b & x > 1 \end{cases},$$

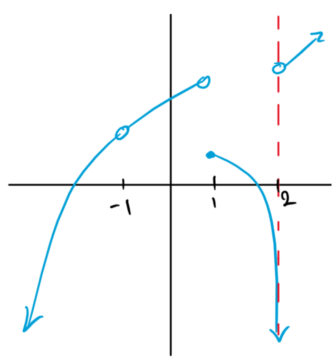
where b is some constant.

(a) Is the function $f(x)$ continuous at $x = 0$? Justify your answer.

(b) Determine the value of b such that f is continuous at $x = 1$.

2. (SFU) Give an example of a function $f(x)$ that is continuous for all values of x except at $x = 3$, where f has a removable discontinuity. Explain how you know f is discontinuous at $x = 3$, and how you know the discontinuity is removable.

3. For the graph below, classify the discontinuities as infinite, jump, or removable.



4. Define a function f by

$$f(x) = \begin{cases} ax + b & x \leq 1 \\ \ln(x) & x > 1 \end{cases}.$$

Find all values of a, b such that f is continuous at $x = 1$.

5. Define a function f by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Show that f is continuous at $x = 0$.

5. Intermediate Value Theorem

1. (SFU)
   (a) Show that $2^x = \frac{10}{x}$ has a solution for some $x > 0$.
   (b) Show that $2^x = \frac{10}{x}$ has no solution for $x < 0$.
2. (SFU) Suppose f is continuous on $[1, 5]$ and the only solution of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.
3. Show that the polynomial $P(x) = x^3 - 2x + 1$ has a root, i.e. $P(c) = 0$ for some $c \in \mathbb{R}$.
4. Let $P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$ be a real polynomial $(a_0, \ldots, a_{n-1} \in \mathbb{R})$ of odd degree (i.e. $n \in \mathbb{N}$ is odd). Use the intermediate value theorem to show that P has at least one real root, i.e. $P(c) = 0$ for some $c \in \mathbb{R}$. *Hint: the argument is similar to Problem 5.3 above.*
5. Let f be a continuous function on some interval $[a, b]$. Prove that $f([a, b])$ is also an interval. *Hint: choose $c, d \in f([a, b])$ and write $c = f(x_1)$, $d = f(x_2)$ for some $x_1, x_2 \in [a, b]$. Since f is continuous on $[x_1, x_2]$, IVT can be applied.*

6. Limits at Infinity, Indeterminate Forms

1. (SFU) Find an example of a function g such that:
   (a) the line $y = 3$ is a horizontal asymptote of the curve $y = g(x)$, **and**
   (b) the curve $y = g(x)$ intersects the line $y = 3$ at infinitely many points.
2. (SFU) Find the vertical and horizontal asymptotes of the curve

$$y = \frac{1 + x^4}{x^2 - x^4}.$$

3. Fix any positive real number $r > 0$. Show that

$$\lim_{x \to \infty} \frac{\ln(x)}{x^r} = 0.$$

4. Determine whether or not the following are indeterminate forms:

$$1^{\infty}, \quad \infty^{1}, \quad \infty^{0}, \quad 0^{\infty}.$$

For the indeterminate forms, give an example of when the limit is 0, 1, and ∞.

5. The formal definition of a limit at infinity is the following:

Given a function f, we say that $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there is $N_\epsilon > 0$ such that if $x > N_\epsilon$, then $|f(x) - L| < \epsilon$.

Use this definition to show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

7. Derivatives I

1. (SFU) Each limit represents the derivative of some function f at some number a. State such an f and a in each case:

(a)

$$\lim_{x\to\frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}.$$

(b)

$$\lim_{x\to 1} \frac{\ln(x)}{x - 1}.$$

(c)

$$\lim_{\theta\to\frac{\pi}{6}} \frac{\sin(\theta) - \frac{1}{2}}{\theta - \frac{\pi}{6}}.$$

2. (SFU) Determine the numbers a, b, c such that the graph of $f(x) = x^3 + ax^2 + bx + c$ satisfies the following condition:

The slope of the secant line defined by the points $P = (0, 1)$ and $Q = (1, 0)$, both on the graph of f, is the slope of the tangent line of f at $x = 1 + \frac{1}{\sqrt{3}}$.

3. (SFU)

(a) Find the asymptotes of the graph of $f(x) = \frac{4-x}{3+x}$ and use them to sketch the graph of f.

(b) Use the graph you found in (a) to sketch the graph of f'.

(c) Use the definition of a derivative to compute $f'(x)$.

4. (SFU)

(a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

5. (SFU) Recall that a function is *even* if $f(x) = f(-x)$ for all x in its domain, and *odd* if $f(-x) = -f(x)$ for all x in its domain. Prove the following **using the definition of the derivative**.

   (a) The derivative of an even function is an odd function.

   (b) The derivative of an odd function is an even function.

8. Derivatives II

1. (SFU) Find values for the constants m and b such that

$$f(x) = \begin{cases} e^x & x \leq 1 \\ mx + b & x > 1 \end{cases}$$

   is continuous and differentiable at $x = 1$.

2. (SFU) Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, $g'(4) = -3$. Find $h'(4)$ when"

   (a) $h(x) = 3f(x) + 8g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \frac{f(x)}{g(x)}$

   (d) $h(x) = \frac{g(x)}{f(x)+g(x)}$.

3. (SFU) How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do the tangent lines touch the curve?

4. (SFU) Differentiate the following functions:

   (a) $f(x) = e^7$.

   (b) $G(t) = \sqrt{5t} + \frac{\sqrt{7}}{t}$.

   (c) $k(r) = e^r + r^e$.

   (d) $f(z) = (1 - e^z)(z + e^z)$.

   (e) $h(r) = \frac{ae^r}{b+e^r}$, where a, b are constants.

   (f) $l(t) = \frac{3t^2+2t-1}{t^3-4}$.

5. Suppose $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$ for all $x \in \mathbb{R}$. Assume that

$$f(1) = 2, \quad f'(1) = 3, \quad g(1) = -1, \quad g'(1) = 4.$$

Let $h(x) = f(x)/g(x)$.

   (a) Determine whether h is increasing or decreasing at $x = 1$.

   (b) Suppose further that f, g satisfy the relation $f(x)g(x) = k$ for some $k \in \mathbb{R}$. In this case, do we need all four pieces of data $f(1), g(1), f'(1), g'(1)$ to determine if h is increasing or decreasing at $x = 1$?

9. Chain Rule, Implicit Differentiation

1. (SFU) Find the derivatives of the following functions:

   (a) $F(x) = (1 + x + x^2)^{101}$.

   (b) $g(x) = (2ra^{rx} + n)^t$.

   (c) $s(t) = \sqrt{\frac{1+\ln(x)}{1+\ln(1/x)}}$.

2. (SFU) If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

3. (SFU) If f and g are two functions for which $f' = g$ and $g' = f$, then show that $f^2 - g^2$ must be a constant function.

4. (SFU) Find dy/dx by implicit differentiation in each of the following cases:

   (a) $x^2 + 4xy + y^2 = 13$.

   (b) $\frac{x^2}{x+2y} = y^3 + 1$.

   (c) $\ln(xy) = 1$.

5. (SFU) Show that the ellipse $x^2 + 2y^2 = 2$ and the hyperbola $2x^2 - 2y^2 = 1$ intersect at right angles.

10. Linear Approximation

1. (SFU) Suppose f satisfies $f(3) = -2$, $f'(3) = -3$, and $f''(3) = 2$.

   (a) Estimate $f(2.95)$ and $f(3.05)$.

   (b) Are your estimates in (a) too small or too large?

   (c) Use differentials to estimate the errors in your estimates.

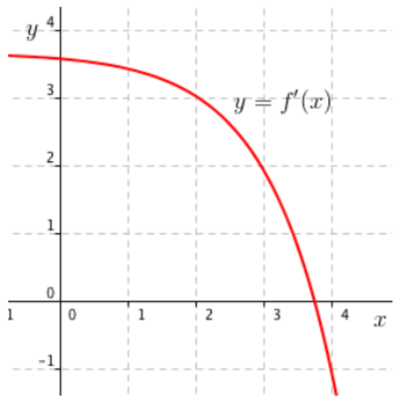
2. (SFU) Let $f(x) = \sqrt{1 + 2x}$.

(a) Find the linear approximation of $f(x)$ at $x = 0$.

(b) Use (a) to estimate the value of $\sqrt{1.1}$.

(c) Is your estimate an overestimate or underestimate?

3. (SFU) Suppose the only information about a function f is that $f(2) = 5$ and the graph of $f'(x)$ **the derivative of** f is shown below:



(a) Use linear approximation to approximate the values of $f(1.9)$ and $f(2.1)$.

(b) Are your estimates in (a) too small or too large? Explain.

4. Let $f(x) = \ln(x)$.

(a) Find the linearization $L(x)$ of $f(x)$ at $x = 1$.

(b) Use (a) to approximate $f(1.1)$. Is this an over or under approximation?

(c) By Taylor's theorem on the interval $[1, 1.1]$, we can write

$$f(x) = f(1) + f'(1)(x - 1) + \varepsilon_2(x)$$

for some error $\varepsilon_2(x)$. Taylor's theorem gives an explicit formula for ε_2,

$$\varepsilon_2(x) = \frac{f''(c)}{2}(x - a)^2$$

for some $c \in (1, 1.1)$.

Use this remainder formula to find an upper bound on the error between f and its linear approximation at $x = 1.1$. That is, find an upper bound of $|\varepsilon_2(1.1)|$.

(d) Use a calculator to compute the error $|f(1.1) - L(1.1)|$ and compare your result to the error bound you found in (c).

5. Show that the linear approximation of a differentiable function f well-approximates f near a point x. More precisely, show that

$$\lim_{h\to 0} \frac{f(x+h) - (f(x) + f'(x)h)}{h} = 0.$$

Hint: Start with the definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This is equivalent to saying that, for small h*, we can write*

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \varepsilon(h)$$

for some function $\varepsilon(h)$ *that satisfies* $\lim_{h\to 0} \varepsilon(h) = 0$.

11. Extrema

1. (SFU) Find the critical points of the functions:
   (a) $g(t) = |2t - 5|$.
   (b) $h(p) = \frac{p-2}{p^4+4}$.
2. (SFU) Find the maximum values of the function $f(x) = x^a(1-x)^b$, given that $a > 1$ and $b > 1$.
3. (SFU) Sketch the graph of a function $f(x)$ that is continuous on $[1, 5]$, has an absolute maximum at $x = 4$, an absolute minimum at $x = 5$, a local maximum at $x = 2$, and a local minimum at $x = 3$.
4. (SFU) Sketch the graph of

$$f(x) = \begin{cases} x^2 & -1 \le x \le 0 \\ 2 - 3x & 0 < x \le 1 \end{cases}$$

   and use your sketch to identify the absolute and local extrema of f.
5. (a) Let f be a differentiable function on (a, b) and suppose $f(c)$ is a local maximum of f for some $c \in (a, b)$. Show that $f'(c) = 0$.

   Hint: since $f(c)$ *is a local maximum of* f*, we have* $f(x) \le f(c)$ *for all* x *close enough to* c*. Use this to study the sign of the difference quotient*

$$\frac{f(x) - f(c)}{x - c}$$

   in the two cases $x < c$ *and* $x > c$*. Then apply the definition of the derivative to conclude that* $f'(c) = 0$.

(b) Use (a) and the Extreme value theorem (EVT) to deduce Rolle's theorem: Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$, then $f'(c) = 0$ for some $c \in (a, b)$.

Hint: If f is the zero function, then Rolle's theorem holds for all $c \in (a, b)$. Assume f is non-zero and apply the EVT.

12. Mean Value Theorem (MVT)

1. (SFU) Verify that the function
$$g(x) = \frac{3x}{x+7}$$
satisfies the hypotheses of the MVT on the interval $[-1, 2]$. Then, find all numbers c that satisfy the conclusion of the MVT (leave your final answer(s) exact).

2. (SFU) Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the MVT?

3. (SFU) Let c be a constant. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

4. Let $f(x) = \ln(x)$. Use the MVT to find rational numbers a, b such that $a \leq \ln(1.1) \leq b$.

5. Use the MVT to show that $1 + x < e^x$ for any $x > 0$.

13. Curve-sketching

1. (SFU) Complete the following questions to sketch the graph of $f(x) = 3x^4 - 8x^3 + 10$.
   (a) Where is the graph increasing, and where is it decreasing?
   (b) Where is the graph concave up, and where is it concave down?
   (c) Where are the local minima and maxima of f. Establish conclusively that they are local extrema.
   (d) Where are the inflection points of f?
   (e) What happens to f as $x \to \infty$ and $x \to -\infty$?
   (f) Use the above to sketch the graph of f.

2. (SFU) Let $f(x) = \frac{x^2-1}{x}$.
   (a) Find the domain of f and the x-intercepts.
   (b) Find all asymptotes of f.

(c) Determine the intervals where f is increasing, and the intervals where f is decreasing. Find the local maxima and minima, if they exist.

(d) Determine the intervals where f is concave up, and the intervals where f is concave down. Find the inflection points, if they exist.

(e) Use the above to sketch the graph of f.

3. (SFU) Let $f(x) = x^{2/3}\left(\frac{5}{2} - x\right)$.

   (a) Explain why f is continuous on $(-\infty, \infty)$.

   (b) Determine the behaviour of f as $x \to \pm\infty$.

   (c) Find $f'(x)$, locate the local minima and maxima of f, and determine the intervals on which f is increasing/decreasing.

   (d) Find $f''(x)$, locate the inflection points of f, and determine the intervals on which f is concave up/down.

   (e) Sketch the graph of f. Clearly indicate and label all intercepts, local extrema, and inflection points.

4. Sketch the graph of $f(x) = x^3 e^{-x+5}$. Clearly indicate and label all intercepts, local extrema, and inflection points.

5. Sketch the graph of $f(x) = 4x^{1/3} + x^{4/3}$. Clearly indicate and label all intercepts, local extrema, and inflection points.