Finite Element Approximation of the Modified Steklov-Maxwell Eigenproblem

by

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Abstract

The problem of determining the spectrum of eigenvalues of the Laplace operator is important for physical applications and mathematical theory. A particular case of these problems are the Steklov eigenvalue problems, where the spectral parameter appears in the boundary condition. We know little about the eigenvalues of vectorial analogs of the Laplace operator, their physical interpretations, or the algorithms used to approximate them. One such analog is the curl-curl operator found in Maxwell's equations, describing the behaviour of electric and magnetic fields. Even formulating a well-defined Steklov eigenvalue problem for the curl-curl operator is challenging due to an eigenvalue of infinite multiplicity. We instead introduce two parameters and study a closely related Steklov eigenvalue problem for Maxwell's equations. We are not aware of any numerical analysis of this modified eigenvalue problem. We provide two approximation approaches based on the finite element method. We describe a series of numerical experiments that examine the convergence of our approximations for specified parameters. Our approach establishes the foundation for further numerical studies of the modified Steklov-Maxwell eigenvalues using finite elements.

Keywords: curl-curl operator; Maxwell's equations; Steklov eigenvalue problems; finite element method; convergence study

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Table of Contents

Declaration of Committee **ii**

Abstract **iii**

Acknowledgements **iv**

Table of Contents **v**

List of Tables **vii**

List of Figures **viii**

1 Introduction **1**

2 The Modified Steklov-Maxwell Eigenproblem **6**

2.1 Function Spaces 6

2.2 The Steklov-Laplace Problem 7

2.3 The Curl-Curl Operator 11

2.4 The Original Steklov-Maxwell Problem 12

2.5 The Weak Modified Steklov-Maxwell Problem 13

2.5.1 Problem Statement 13

2.5.2 Strong Implies Weak 14

2.5.3 Weak Implies Strong, with Assumptions 15

2.6 Properties of the Modified Steklov-Maxwell Problem 16

3 The Discrete Modified Steklov-Maxwell Eigenproblem **18**

3.1 Finite Element Method 18

3.1.1 The Characterization Theorem 18

3.1.2 Finite Elements 19

3.1.3 A Two-Dimensional Example 21

3.1.4 Matrices and Approximation 22

3.1.5 The Discrete Steklov-Maxwell Problem 22

3.2 Numerically Determining the Eigenvalues 23

3.2.1 Shift-Invert Method . . . 24
3.2.2 Power Iteration . . . 24
3.2.3 Arnoldi Iteration . . . 25
3.2.4 Condition Numbers . . . 26
3.3 Numerical Results . . . 27
3.3.1 Results for the Steklov-Laplace Problem . . . 27
3.3.2 Nonconforming Method . . . 28
3.3.3 Conforming Method . . . 30

4 Conclusion **33**

Bibliography **35**

List of Tables

Table 3.1 Condition numbers of the finite element matrices. 29
Table 3.2 Eigenvalues for the ball compared with eigenvalues on the cube. 31
Table 3.3 Slopes of the best-fit lines corresponding to convergence data. 32

List of Figures

Figure 1.1 Diagrammatic thesis outline. 5

Figure 3.1 Two meshes approximating the unit sphere. 20
Figure 3.2 An example mesh in two dimensions. 21
Figure 3.3 Convergence plot for the Steklov-Laplace problem on the unit disk. 28
Figure 3.4 True eigenvalues on the unit ball compared to approximate eigenvalues. 29
Figure 3.5 Convergence plot for the nonconforming method. 30
Figure 3.6 Convergence plots for the conforming method. 32

Chapter 1

Introduction

In his 1966 article, Kac [9] asked "Can one hear the shape of the drum?" to explore the relation between the structure of a domain and the eigenvalues of its differential operators. The answer provides a beautiful relationship between linear algebra and partial differential equations. In particular, we arrive at an eigenvalue problem for the Laplace operator (commonly called the Laplacian). The eigenvalues are intimately connected to the shape of the domain and the prescribed boundary conditions.

A particularly intriguing eigenvalue problem is the *Steklov eigenvalue problem for the Laplacian*, frequently referred to as the *Steklov-Laplace problem*. Let

$$\mathbb{R}^d = \{(x_1, \ldots, x_d) : x_i \in \mathbb{R},\ i = 1, \ldots, d\}.$$

Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega$. Let $\mathbf{a} \cdot \mathbf{b}$ denote the dot product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$. As described by Levitin, Mangoubi, and Polterovich in [14], the Steklov-Laplace problem is to find pairs consisting of a non-zero function u and a constant σ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = \sigma u & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\nabla := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ is the gradient operator on Ω, $\mathbf{n}$ is the outward unit normal vector on Γ, and $\Delta u := \nabla \cdot \nabla u$ is the Laplacian on Ω. We call any pair (σ, u) solving (1.1) an *eigenpair*, where u is an *eigenfunction* with corresponding *eigenvalue* σ. The set of all eigenvalues of an eigenvalue problem is called its *spectrum*. Vladimir Steklov formulated the problem (1.1) around the year 1900, and it is one of his significant contributions to mathematical physics. The eigenvalues of (1.1) are closely related to the sloshing problem: how does one's morning coffee spill over when rushing out the door to get to work? In particular, the Steklov boundary condition in (1.1) is present in the sloshing problem; by understanding the Steklov eigenvalues of the Laplacian, we can better understand the sloshing behaviour of a fluid as the eigenvalues are frequencies describing the oscillations of the fluid in its cup. Namely, the derivative of u near Γ becomes large as σ increases, so the oscillations of u localize near the boundary of the domain. Kuznetsov et al. [10] provide a fascinating overview of this result, along with a detailed history of problem (1.1). For a thorough summary of the Steklov-Laplace problem and related open problems, we refer the reader to [5].

We call (1.1) the *strong formulation* of the Steklov-Laplace problem. While solving the strong form of an eigenvalue problem is essential for describing various physical phenomena, it is convenient to work instead with a *weak formulation.* As an example, let us define the weak formulation of (1.1). Firstly, let $H^1(\Omega)$ denote the first *Sobolev space*, consisting of functions whose derivatives are "well-behaved" (which we will define more precisely in Chapter 2). The weak formulation of the Steklov-Laplace problem is to find $u \in H^1(\Omega)$ and $\sigma \in \mathbb{C}$ such that for all $v \in H^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v \ \mathrm{d}x = \sigma \int_{\Gamma} uv \ \mathrm{d}s. \tag{1.2}$$

Throughout this paper, dx and ds are integration elements corresponding to volume integrals and surface integrals, respectively. See [14] for a derivation of (1.2).

Solutions of the strong and weak formulations of an eigenvalue problem are called strong and weak solutions, respectively. Strong solutions are always weak solutions, and the converse typically holds under problem-specific regularity assumptions. Essentially, the function space containing the strong solutions is a proper subset of the space containing the weak solutions. Thus, the weak and strong formulations of an eigenvalue problem are intimately connected. Since weak problems may possess solutions even when strong solutions do not exist, these formulations are invaluable in numerical analysis. For example, given a weak solution (σ, u) of (1.2), if we attempt to show that (σ, u) solves the strong problem, we will quickly realize that $u \in H^1(\Omega)$ is not sufficient: we need better-behaved derivatives than $H^1(\Omega)$ provides. Regardless, a comprehensive numerical study of a weak problem often produces valuable insights into the strong problem's behaviour.

Notice that the Steklov-Laplace problem involves a scalar field u. However, numerous differential operators act on vector fields, so we can generalize the problem to the vectorial case. For instance, a natural analog of the Laplacian for vector fields is the curl-curl operator, or the "curl of a curl", defined as

$$\operatorname{curl} \operatorname{curl} \mathbf{u} \coloneqq \nabla \times (\nabla \times \mathbf{u}),$$

where ∇ is the gradient operator on $\mathbb{R}^3$, $\mathbf{u} = (u_1(x,y,z), u_2(x,y,z), u_3(x,y,z))$ is a differentiable vector field, and $\times$ denotes the cross product on $\mathbb{R}^3$. The curl of a vector field, $\operatorname{curl} \mathbf{u} \coloneqq \nabla \times \mathbf{u}$, is a vector that measures the tendency of $\mathbf{u}$ to rotate. Therefore, $\operatorname{curl} \operatorname{curl} \mathbf{u}$ measures the tendency of $\operatorname{curl} \mathbf{u}$ to rotate, or how the rotation of $\mathbf{u}$ rotates at a point. Studies of the curl-curl operator apply to various areas of physics, such as electromagnetics, the theory of superconductors, and magnetohydrodynamics; it is beneficial to investigate the curl-curl operator's behaviour with different boundary conditions [11].

Formulating a meaningful generalization of the Steklov-Laplace problem (1.1) to vectorial problems is not a trivial task, as we will see shortly. One attempt at generalizing the problem to the curl-curl operator is to look for vector fields $\mathbf{u}$ and scalars $\lambda \in \mathbb{C}$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u} = 0 & \text{in } \Omega, \\ \nu \times (\operatorname{curl}\mathbf{u}) = \lambda\mathbf{u} & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with sufficiently regular boundary $\Gamma = \partial\Omega$, and ν is the unit outward normal vector on Γ. Concretely, Γ is of class $C^{1,1}$ (as defined in [12]). However, there is a major issue with this formulation. Namely, if $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable scalar function, then $\operatorname{curl}\operatorname{grad}\phi = 0$, where the vector field $\operatorname{grad}\phi \coloneqq \nabla\phi$ is the gradient of ϕ. Thus, if $\mathbf{u}$ is any gradient, then the eigenpair $(0, \mathbf{u})$ satisfies (1.3). In other words, the eigenvalue $\lambda = 0$ has infinite multiplicity, and the other eigenvalues accumulate only at infinity. If eigenvalues represent frequencies, then determining the non-zero frequencies is vital. But since $\lambda = 0$ has infinite multiplicity, the linear system we use to approximate the eigenvalues of (1.3) consists of only zero eigenvalues. Consequently, we must modify the problem and instead study a related problem with properly behaved eigenvalues.

The modified Steklov-Maxwell eigenvalue problem, first introduced by Lamberti & Stratis in [11], is to find vector fields $\mathbf{u}$ and constants λ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} - \theta\operatorname{grad}\operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \nu \times (\operatorname{curl}\mathbf{u}) = \lambda\mathbf{u} & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $\alpha \in \mathbb{R}$ and $\theta > 0$ are parameters, and $\operatorname{div}\mathbf{u} \coloneqq \nabla \cdot \mathbf{u}$ is the divergence of u. The parameters α and θ act to shift the eigenvalues λ of (1.4) away from 0, so we call $\alpha\mathbf{u}$ and $\theta\operatorname{grad}\operatorname{div}\mathbf{u}$ *regularization terms*. This variant of the problem stems from the time-harmonic form of Maxwell's equations,

$$\operatorname{curl}\mathbf{E} - i\omega\mu\mathbf{H} = 0, \ \operatorname{curl}\mathbf{H} + i\omega\epsilon\mathbf{E} = 0,$$

where $\mathbf{E}$ and $\mathbf{H}$ are the spatial components of an electric and magnetic field, respectively, and ω, μ, and ϵ are constants.

Camaño, Monk & Lackner in [3] formulated the original Steklov-Maxwell eigenvalue problem for use in inverse scattering theory. However, the spectrum of the unmodified problem on the unit ball consists of two sequences of eigenvalues, one tending to 0 and the other to infinity. Thus, Lamberti & Stratis proposed (1.4) as an alternative problem that exhibits just one sequence of eigenvalues that monotonically decreases to negative infinity. The spectrum of the original problem and (1.4) is *discrete* in the sense that each point is isolated. Furthermore, each eigenvalue of (1.4) is of finite multiplicity. While other modifications have been proposed [3], [6], (1.4) has the advantage of being easily discretized for numerical studies.

The primary goal of this work is to develop a numerical framework for studying the modified Steklov-Maxwell problem. In particular, we apply the finite element method to approximate the eigenvalues of (1.4). It is of principal importance in numerical analysis to characterize the consistency, stability, and convergence of a given algorithm, which impacts robustness and efficiency. Thus, we conduct experiments to numerically study the convergence of the finite element method for our problem. We also examine how sensitive the corresponding matrices are to small changes in their data.

Chapter 2 presents the modified Steklov-Maxwell problem and its theoretical properties. We begin by briefly explaining the properties of the Steklov-Laplace problem and precisely restating its weak formulation. In doing this, we build the theoretical framework to generalize the problem to the curl-curl operator and discuss the corresponding issues with our generalization. Subsequently, we present the modified Steklov-Maxwell problem and derive its weak and variational forms. Lastly, we summarize the work of Lamberti & Stratis in [11] and Ferraresso, Lamberti, & Stratis in [4] as their research effectively builds the theoretical framework of the problem.

In Chapter 3, we formulate the discrete form of the problem to apply the finite element method. We begin by introducing the finite element method from an abstract perspective. We then define the discrete form of the modified Steklov-Maxwell problem, discuss the corresponding nonconforming and conforming methods for approximating its solutions, and detail the implementation of the algorithm using FreeFem++ [7] (a finite element method software built in C++) and MATLAB [8]. The final section compares nonconforming and conforming methods to approximate the Steklov-Maxwell eigenvalues. Our computations show convergence in both cases, though the conforming approach demonstrates better convergence than the nonconforming approach on the cube. We conclude that the conforming method, while involving ill-conditioned matrices, effectively computes solutions to (1.4). The following diagram illustrates the structure of the thesis. We encourage the reader to periodically refer back to Figure 1.1 for a visual view of how the main elements of this thesis are related.

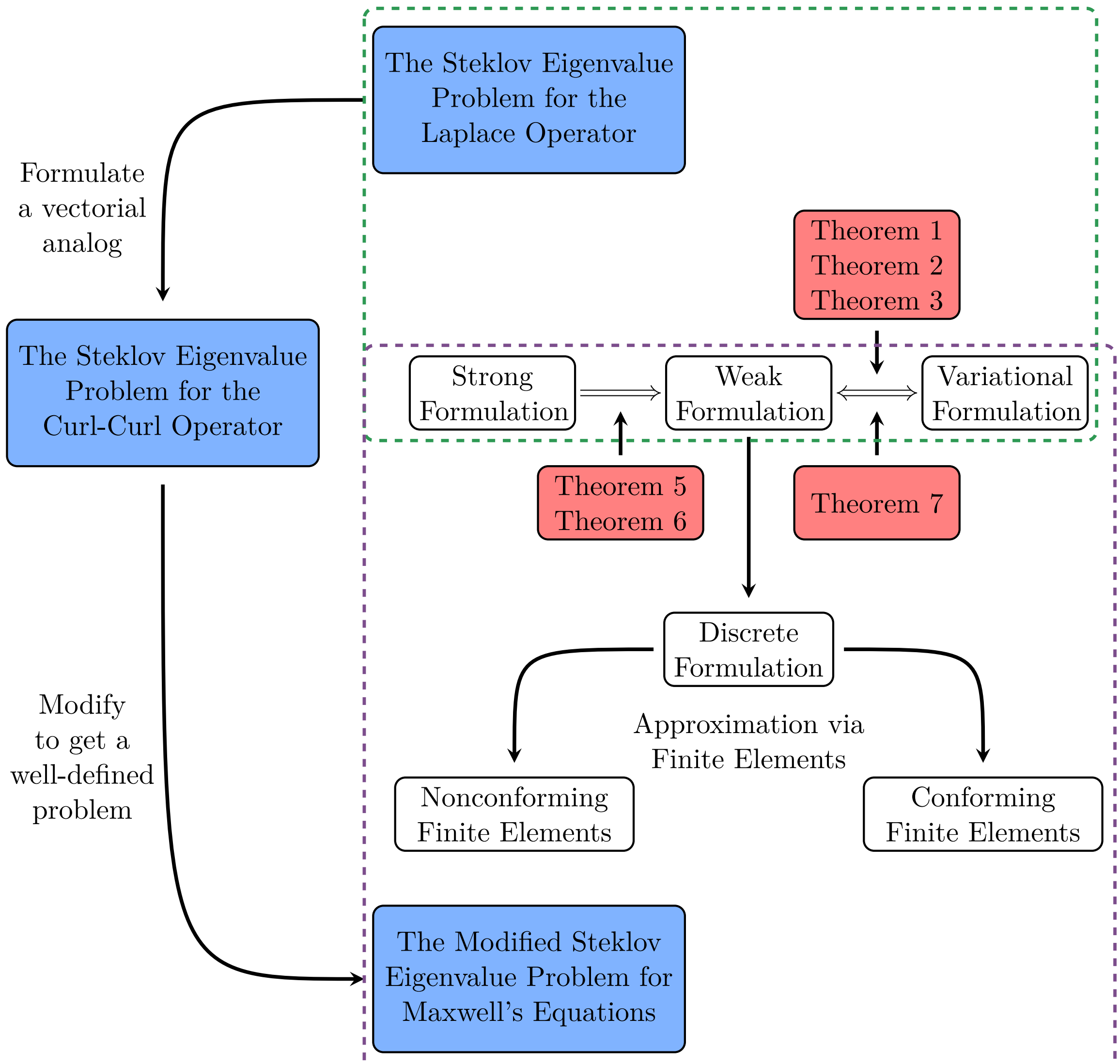


Figure 1.1: A flow chart demonstrating the technical elements of the thesis and their relationships. Bold arrows indicate the flow of the thesis and relationships between the elements and double-lined arrows indicate logical implications. The main concepts related to the Steklov-Laplace and modified Steklov-Maxwell problems are grouped by the dashed-border boxes.

Chapter 2

The Modified Steklov-Maxwell Eigenproblem

2.1 Function Spaces

The definitions in this section are from [14, Chapter 2] and [14, Appendix B].

In Chapter 1, we briefly introduced the strong and weak formulations of the Steklov-Laplace eigenvalue problem, equations (1.1) and (1.2), respectively. We noted that strong solutions are always weak solutions, but the converse is not necessarily true; the weak solutions lie in a larger space than the strong solutions. These spaces are the Sobolev spaces, which arise when we weaken our notion of differentiability. By reinterpreting the meaning of differentiation and constructing the Sobolev spaces, powerful techniques from functional analysis can be applied to develop a general theory for studying eigenvalue problems.

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Roughly, Γ is locally the graph of a Lipschitz function. We refer the reader to [14, Appendix B Section 3] for a rigorous definition. For each $k \in \mathbb{N}$, let $C^k(\Omega)$ denote the set of all real-valued, k times continuously differentiable functions on Ω. Let $C_0^k(\Omega) = \{f \in C^k(\Omega) : \operatorname{supp}(f) \text{ is compact}\}$, where $\operatorname{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$ is the *closed support* of f. Denote by $L^2(\Omega)$ the space of square-integrable and Lebesgue measurable functions[1], equipped with the inner product

$$\langle f, g\rangle_{L^2(\Omega)} \coloneqq \int_\Omega f(x)g(x)\,\mathrm{d}x,$$

for all $f, g \in L^2(\Omega)$. With this inner product, all Cauchy sequences converge. Hence, we call $L^2(\Omega)$ a *Hilbert space.* This inner product induces a norm

$$\|f\|_{L^2(\Omega)}^2 \coloneqq \langle f, f\rangle_{L^2(\Omega)} = \int_\Omega f(x)^2\,\mathrm{d}x.$$

[1]Strictly speaking, $L^2(\Omega)$ consists of equivalence classes of functions such that two functions belong to the same equivalence class if and only if they differ by a set of Lebesgue measure zero.

Define $L^1_{\text{loc}}(\Omega)$ to be the set of all Lebesgue measurable functions that are integrable on any compact subset of Ω. With these spaces, we can define the Sobolev spaces.

Let $u, v \in L^1_{\text{loc}}(\Omega)$ and define $\partial_j := \frac{\partial}{\partial x_j}$. We say that $v = \partial_j u$ in the *weak sense* if for every $\varphi \in C^1_0(\Omega)$, we have

$$\int\limits_{\Omega} u \partial_j \varphi \, \mathrm{d}x = -\int\limits_{\Omega} \varphi \partial_j v \, \mathrm{d}x.$$

In this case, we call v the *weak derivative* of u in x_j. The Sobolev spaces are defined recursively using weak derivatives as follows. Let $H^0(\Omega) = L^2(\Omega)$. For each $k \in \mathbb{N}$, the k-th *Sobolev space*, denoted $H^k(\Omega)$, is defined as all those $u \in L^2(\Omega)$ such that $\partial_j u$ exists in the weak sense and $\partial_j u \in H^{k-1}(\Omega)$. These sets become Hilbert spaces when equipped with the inner products

$$\langle f, g \rangle_{H^k(\Omega)} := \langle f, g \rangle_{L^2(\Omega)} + \sum_{j=1}^{d} \langle f, g \rangle_{H^{k-1}(\Omega)}, \tag{2.1}$$

where $f, g \in H^k(\Omega)$. Note that this recursive definition of the inner product is well-defined as the Sobolev spaces form an increasing sequence of sets. This definition clarifies what we meant in Chapter 1 by functions in $H^1(\Omega)$ having "well-behaved" derivatives.

2.2 The Steklov-Laplace Problem

Since the modified Steklov-Maxwell problem is a vectorial generalization of the Steklov-Laplace problem, we first summarize some noteworthy results concerning the latter. This section also illustrates some indispensable results of the finite element method applied to eigenvalue problems (as discussed briefly in Chapter 3). The theorems in this section are adapted from [14, Chapter 7] and [15, Chapter 11]. The proof techniques used in this section are motivated by those in [15].

Let $\nabla := (\partial_{x_1}, \dots, \partial_{x_d})$ be the gradient operator on $\mathbb{R}^d$. We begin by explaining the significance of the Steklov boundary condition $\nabla u \cdot \mathbf{n} = \sigma u$ on Γ, where Γ is the (sufficiently smooth) boundary to some domain $\Omega \subseteq \mathbb{R}^d$ and $\mathbf{n}$ the outward unit normal vector to Γ. We note that the Steklov-Laplace eigenvalues form a sequence $0 = \sigma_1 < \sigma_2 \leq \dots$ increasing without bound, so we may choose σ to be very large. Since $\nabla u \cdot \mathbf{n}$ is the derivative of u along the outward unit normal to Γ, the Steklov boundary condition $\nabla u \cdot \mathbf{n} = \sigma u$ indicates that u is very steep near Γ whenever σ is large. This feature is the essence of Steklov boundary conditions: as the eigenvalues increase, the oscillations of the corresponding eigenfunctions become localized near the boundary. With this new understanding of the Steklov-Laplace problem, we now describe its various formulations and their connections.

In Chapter 1, we noted the benefits of working with a weak formulation. For instance, this form is convenient for proving various theorems or setting up a "discrete" version of the problem for numerical studies. However, there is one more formulation we consider called the *variational formulation*, where we characterize the Steklov-Laplace eigenvalues as solutions to an optimization problem. This version of the problem is useful for proving numerous important results. See [5] for a comprehensive overview of these results.

Recall, the weak formulation of the Steklov-Laplace problem (1.1) is to find $u \in H^1(\Omega)$ and $\sigma \in \mathbb{C}$ such that for all $v \in H^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x = \sigma \int_{\Gamma} uv \, \mathrm{d}s. \tag{2.2}$$

The elements of $H^1(\Omega)$ are commonly referred to as *test functions*. We show that solving (2.2) is equivalent to solving an optimization problem, the solutions of which we call *variational solutions*. The relationships between the strong, weak, and variational solutions to the Steklov-Laplace problem are summarized in the following chain of implications:

$$\text{Strong Solution} \Longrightarrow \text{Weak Solution} \Longleftrightarrow \text{Variational Solution.}$$

This section is devoted to proving the equivalence between the weak and variational solutions.

We prove the above equivalence in three steps. We first characterize the first non-zero Steklov-Laplace eigenvalue, σ_2, as a solution to a minimization problem (Theorem 1). We then generalize this argument to the n-th eigenvalue, σ_n, though the minimization problem in this case has additional constraints (Theorem 2). Finally, we state the variational formulation of the Steklov-Laplace problem and use the previous two results to prove its equivalence to the weak formulation (Theorem 3).

Denote the Steklov-Laplace eigenvalues on Ω by $0 = \sigma_1(\Omega) < \sigma_2(\Omega) < \dots$ where the $\sigma_k(\Omega)$ increase without bound. Note that the eigenfunction corresponding to $\sigma_1(\Omega)$ is a constant, so the following theorem characterizes the first non-zero eigenvalue, $\sigma_2(\Omega)$, as the solution to a minimization problem.

Theorem 1 (Minimum Principle for the 2nd Steklov-Laplace Eigenvalue)**.** *Let*

$$Q(w) = \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Gamma)}}$$

for all $w \in H^1(\Omega)$. Then $\sigma_2(\Omega)$ is the minimum value of $Q(w)$ over all non-zero test functions $w \in H^1(\Omega)$ satisfying

$$\int_{\Gamma} w \, ds = 0.$$

Furthermore, any minimizing value of $Q(w)$ is an eigenfunction corresponding to $\sigma_2(\Omega)$.

Proof. Suppose u solves the minimum problem and let m be the minimum value. Since Q is non-negative, so is m. Consider any test function $v \in H^1(\Omega)$ and let $\epsilon > 0$. Define

$J(\epsilon) = Q(u+\epsilon v)$. Then J attains a minimum at $\epsilon = 0$. Applying the quotient rule to J and simplifying, we find that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = m \int_\Gamma uv \, \mathrm{d}s.$$

Since v is arbitrary, u is a solution of (2.2).

To show that $m = \sigma_2(\Omega)$, let v_j be the j-th eigenfunction of the Steklov-Laplace problem with corresponding eigenvalue $\sigma_j(\Omega)$. Then $m \leq Q(v_j) = \sigma_j(\Omega)$ by (2.2). □

We can generalize Theorem 1 to the n-th eigenvalue as follows.

Theorem 2 (Minimum Principle for the n-th Steklov-Laplace Eigenvalue)**.** *The n-th Steklov-Laplace eigenvalue $\sigma_n(\Omega)$ is the minimum of $Q(w)$ over all non-zero $w \in H^1(\Omega)$ such that w is orthogonal (in the L^2 sense) to the first $n-1$ eigenfunctions $v_1, \ldots, v_{n-1}$.*

Proof. The $n = 1$ case is immediate as the eigenfunction is constant, and the $n = 2$ case is Theorem 1. We may assume the Steklov-Laplace eigenfunctions are pairwise orthogonal.

Let $n \geq 3$ and suppose m is the minimum value of Q as in the theorem statement. Let u be the minimizing value of Q corresponding to m. The proof showing m is an eigenvalue of (2.2) is identical to the proof presented in Theorem 1. We show $m = \sigma_n(\Omega)$.

Let $k \geq n$ and let v_k denote the k-th Steklov-Laplace eigenfunction. Then

$$m = Q(u) \leq Q(v_k) = \sigma_k$$

by the minimizing properties of u and (2.2). Also, the problem of minimizing Q over all non-zero w orthogonal to the first $n-1$ eigenfunctions has more constraints than the minimization problem over all non-zero w orthogonal to the first $n-2$ eigenfunctions. Thus, if m' minimizes the latter problem, then $m' \leq m$. By induction, $m' = \sigma_{n-1}$, so we have that $\sigma_{n-1} \leq m \leq \sigma_n$. It follows from $u \neq v_{n-1}$ that $m = \sigma_n$. □

We use Theorem 2 to provide an elementary proof of the variational Steklov-Laplace problem. Our proof contrasts with the proof presented in [14], which uses a direct sum of two function spaces and the Dirichlet-Laplace eigenvalue problem.

Theorem 3 (The Variational Formulation of the Steklov-Laplace Problem)**.**
For each $n \in \mathbb{N}$,

$$\sigma_n(\Omega) = \min_{\substack{V \subseteq H^1(\Omega) \\ \dim V = n}} \max_{\substack{w \in V \\ w \not\equiv 0}} Q(w). \tag{2.3}$$

Proof. Let $\mathcal{B} = \{w_1, \ldots, w_n\}$ be a set of n linearly independent functions in $H^1(\Omega)$, and let $V = \operatorname{span}(\mathcal{B})$. Choose $c_1, \ldots, c_n \in \mathbb{R}$ not all zero such that $w = c_1 w_1 + \ldots + c_n w_n$ is orthogonal to the first $n-1$ Steklov-Laplace eigenfunctions. Such a choice can be made as this amounts to solving an $(n-1) \times n$ linear system for the zero vector, which always has a non-trivial solution. By Theorem 2, $\sigma_n \leq Q(w) \leq \max Q(w)$, where we take the maximum over all possible linear combinations of the basis elements. Since $\mathcal{B}$ is arbitrary, we may take the minimum on each side to get

$$\sigma_n(\Omega) \leq \min_{\substack{V \subseteq H^1(\Omega) \\ \dim V = n}} \max_{\substack{w \in V \\ w \not\equiv 0}} Q(w).$$

Conversely, let $w_i = v_i$, for each $i = 1, \ldots, n$, where v_i is the i-th eigenfunction. We may assume the eigenfunctions are pairwise orthogonal and normalized on the boundary: for all $i = 1, \ldots, n$ we have $\|v_i\|^2_{L^2(\Gamma)} = 1$. Let $V = \operatorname{span}(\{v_1, \ldots, v_n\})$. Then each non-zero $w \in V$ can be represented as $w = c_1 v_1 + \ldots + c_n v_n$ for some non-zero vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$. Since the v_k are pairwise orthogonal and each is normalized on Γ,

$$\begin{aligned}
Q(w) = \frac{\|\sum_{i=1}^n c_i \nabla v_i\|^2_{H^1(\Omega)}}{\|\sum_{i=1}^n c_i v_i\|^2_{H^1(\Gamma)}} &= \frac{\int_\Omega \left(\sum_{i=1}^n c_i \nabla v_i\right) \cdot \left(\sum_{i=1}^n c_i \nabla v_i\right) \, \mathrm{d}x}{\int_\Gamma \left(\sum_{i=1}^n c_i v_i\right) \cdot \left(\sum_{i=1}^n c_i v_i\right) \, \mathrm{d}s} \\
&= \frac{\sum_{i=1}^n c_i^2 \int_\Omega |\nabla v_i|^2 \, \mathrm{d}x}{\sum_{i=1}^n c_i^2 \int_\Gamma |v_i|^2 \, \mathrm{d}s} \\
&= \frac{\sum_{i=1}^n c_i^2 \sigma_i}{\sum_{i=1}^n c_i^2} \\
&\leq \frac{\sum_{i=1}^n c_i^2 \sigma_n}{\sum_{i=1}^n c_i^2} \\
&= \sigma_n.
\end{aligned}$$

Since w was arbitrarily chosen, $\sigma_n = \max_w Q(w)$ for our particular choice of V. Hence,

$$\min_{\substack{V \subseteq H^1(\Omega) \\ \dim V = n}} \max_{\substack{w \in V \\ w \not\equiv 0}} Q(w) \leq \sigma_n(\Omega),$$

proving the claim. □

In particular, Theorem 3 says that (2.2) is equivalent to (2.3), so determining the weak Steklov-Laplace eigenpairs is equivalent to solving an optimization problem. We now have the freedom to work with whichever formulation is most convenient. In general, the variational formulation (2.3) is a powerful tool for proving geometric inequalities involving the eigenvalues. Determining such inequalities is an active area of research for the Steklov-Laplace problem, as we can rarely find the eigenvalues exactly.

We conclude this section with the statement of a special scaling property of the eigenvalues. For $c > 0$ given, the set $\Omega_c = \{cx : x \in \Omega\}$ is called a *homothety* of Ω and c the *homothety ratio*.

Theorem 4 (Homothety Property of the Steklov-Laplace Problem)**.** *Let $c > 0$ and let Ω_c be a homothety of Ω. Then for all $n \in \mathbb{N}$,*

$$\sigma_n(\Omega) = c\sigma_n(\Omega_c).$$

Namely, determining the eigenvalues for the domain Ω tells us the eigenvalues on any scalar multiple of Ω. The theorem follows from a change of variables in the variational formulation and some simplification. We will generalize Theorem 4 to the modified Steklov-Maxwell eigenvalue problem in Section 2.6, though the statement becomes more complicated.

2.3 The Curl-Curl Operator

The Steklov-Laplace problem is very well-studied [5]. Thus, one may wonder how the problem generalizes to operators that act on vector fields. One such operator is the curl-curl operator, which, like the Laplace operator, is a second-order differential operator. In Chapter 1, we briefly saw the failure of the natural analog of the Steklov-Laplace problem to the curl-curl operator. We expand on these ideas in this section.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with sufficiently regular boundary $\Gamma = \partial\Omega$. Let ν denote the outward unit normal to Γ. As in Chapter 1, the curl operator is the cross product $\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u}$. Based on the strong Steklov-Laplace problem, the candidate Steklov eigenproblem for the curl-curl operator is to find vector fields $\mathbf{u}$ and constants $\lambda \in \mathbb{C}$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl} \mathbf{u} = 0 & \text{in } \Omega, \\ \nu \times (\operatorname{curl} \mathbf{u}) = \lambda \mathbf{u} & \text{on } \Gamma. \end{cases} \tag{2.4}$$

The immediate issue with (2.4) is that any gradient solves the problem with $\lambda = 0$, so the eigenspace corresponding to $\lambda = 0$ has infinite multiplicity. Hence, the problem is not well-defined, as our general theory of eigenvalue problems requires each eigenvalue to be of finite multiplicity. Furthermore, a numerical study of the eigenvalues of (2.4) is impossible as the linear system we use for our approximations has only zero eigenvalues. That is, we cannot numerically study the non-zero eigenvalues. However, we can still say something about the eigenfunctions corresponding to non-zero eigenvalues of (2.4).

Remark. *If λ is a non-zero eigenvalue of (2.4) and $\mathbf{u}$ a corresponding eigenfunction, then $\mathbf{u}$ is tangential. That is, $\nu \cdot \mathbf{u} = 0$ on Γ.*

Proof. Since $\mathbf{u}$ is an eigenfunction of (2.4) with corresponding eigenvalue $\lambda \in \mathbb{C}$, then it satisfies the boundary condition $\nu \times (\operatorname{curl} \mathbf{u}) = \lambda \mathbf{u}$ on Γ. Taking the dot product of the boundary condition with ν gives $\lambda(\nu \cdot \mathbf{u}) = 0$, as $\nu \times (\operatorname{curl} \mathbf{u})$ is orthogonal to ν. However, $\lambda \neq 0$, so $\nu \cdot \mathbf{u} = 0$ on Γ, as desired. □

2.4 The Original Steklov-Maxwell Problem

We introduced (1.4) as the *modified* Steklov-Maxwell problem because it arises from a slight modification to the original problem formulated by Camaño, Monk, & Lackner in [3]. Before studying (1.4), we introduce the original Steklov-Maxwell problem and briefly discuss the necessity of modifying it.

Let us take Ω to be the unit ball in $\mathbb{R}^3$, bounded by the unit sphere $\Gamma = \partial\Omega$. For a vector field $\mathbf{w}$ on Ω, its tangential component is defined as $\mathbf{w}_T := \nu \times \mathbf{w} \times \nu$, where ν is the outward unit normal of Ω. The Steklov-Maxwell eigenvalue problem is to find vector fields $\mathbf{w}$ and constants $\lambda \in \mathbb{C}$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{w} - \alpha\mathbf{w} = 0 & \text{in } \Omega, \\ \nu \times \operatorname{curl}\mathbf{w} = \lambda\mathbf{w}_T & \text{on } \Gamma, \end{cases} \tag{2.5}$$

where $\alpha \in \mathbb{R}$ is a constant that is not an eigenvalue of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{v} = \alpha\mathbf{v} & \text{in } \Omega, \\ \nu \times \operatorname{curl}\mathbf{v} = 0 & \text{on } \Gamma. \end{cases}$$

Problem (2.5) is shown in [3] to have infinitely many eigenvalues on the unit ball, forming a real-valued sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ monotonically increasing to infinity.

Assume $\alpha \neq 0$. Define $\mathbf{u} := 1/(i\sqrt{\alpha})\operatorname{curl}\mathbf{w}$, where $i = \sqrt{-1}$ and $\mathbf{w}$ is an eigenvector of (2.5) with corresponding eigenvalue $\lambda \neq 0$. Employing various vector identities gives that $\mathbf{u}$ is an eigenvector of (2.5) corresponding to the eigenvalue $-\alpha/\lambda$. That is,

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} = 0 & \text{in } \Omega, \\ \nu \times \operatorname{curl}\mathbf{u} = -\frac{\alpha}{\lambda}\mathbf{u}_T & \text{on } \Gamma. \end{cases}$$

In particular, $\{-\alpha/\lambda_n\}_{n\in\mathbb{N}}$ is another sequence of eigenvalues of (2.5) on Ω, and $-\alpha/\lambda_n \to 0$ as $n \to \infty$. Thus, assuming $\lambda_1 \neq 0$, there is a second sequence of eigenvalues that converges to zero. This is an obstacle as our theory of eigenvalue problems relies on the problem having a single sequence of eigenvalues increasing or decreasing without bound. Furthermore, the linear operator corresponding to (2.5) is not compact (a property crucial for well-defined eigenvalue problems). See [3] for the complete details of the original Steklov-Maxwell eigenvalue problem. The modification proposed by Lamberti & Stratis in [11], problem (1.4), corrects both of these issues.

While (2.5) is the motivating problem for the work of Lamberti & Stratis, our interests lie in formulating a vectorial analog of the Steklov-Laplace problem (1.1). Thus, our approach emphasizes studying the modified Steklov-Maxwell problem in the context of determining a well-defined Steklov eigenvalue problem for the curl-curl operator. However, the issues with both the original Steklov-Maxwell problem (2.5) and our attempt at a Steklov-Curl-Curl problem (2.4) demonstrate that it is not straightforward to formulate and study vectorial Steklov eigenvalue problems.

2.5 The Weak Modified Steklov-Maxwell Problem

2.5.1 Problem Statement

A prerequisite to establishing the numerical framework of the Steklov-Maxwell problem (1.4) is a better understanding of its theoretical foundation. As the modified Steklov-Maxwell problem is a vectorial analog of the Steklov-Laplace problem, we wish to generalize the latter's properties to the former. However, we first need a weak formulation of the Steklov-Maxwell problem. In this chapter, we assume $\Omega \subseteq \mathbb{R}^3$ is a bounded domain of class $C^{1,1}$ (as defined in [12]). That is, the boundary of Ω, written $\Gamma = \partial\Omega$, is sufficiently regular. Unless otherwise stated, the definitions and results of this section are adapted from Lamberti & Stratis in [11].

Just as we needed the Sobolev spaces for the Steklov-Laplace problem, we also need vectorial analogs of these spaces for the modified Steklov-Maxwell problem. We define the vectorial Sobolev spaces and their norms as

$$\begin{aligned} H(\mathrm{curl}, \Omega) &\coloneqq \{\mathbf{u} \in (L^2(\Omega))^3 : \mathrm{curl}\,\mathbf{u} \in (L^2(\Omega))^3\}, \\ \|\mathbf{u}\|_{H(\mathrm{curl},\Omega)} &\coloneqq \|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\mathrm{curl}\,\mathbf{u}\|^2_{(L^2(\Omega))^3}, \\ H(\mathrm{div}, \Omega) &\coloneqq \{\mathbf{u} \in (L^2(\Omega))^3 : \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\}, \\ \|\mathbf{u}\|_{H(\mathrm{div},\Omega)} &\coloneqq \|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\mathrm{div}\,\mathbf{u}\|^2_{L^2(\Omega)}, \end{aligned}$$

where $\mathrm{div}\,\mathbf{u} = \nabla \cdot \mathbf{u}$ is the divergence of $\mathbf{u}$. Also define

$$H_0(\mathrm{div}, \Omega) \coloneqq \{\mathbf{u} \in H(\mathrm{div}, \Omega) : \nu \cdot \mathbf{u} = 0 \text{ on } \Gamma\}, \tag{2.6}$$

where ν is the unit outward normal on Γ. Any vector field $\mathbf{u}$ satisfying $\mathbf{u} \cdot \nu = 0$ is called *tangential*. The normed space of test functions we work in is $X_{\mathrm{T}}(\Omega)$, defined as

$$\begin{aligned} X_{\mathrm{T}}(\Omega) &\coloneqq H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega), \qquad (2.7) \\ \|\mathbf{u}\|^2_{H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)} &\coloneqq \|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\mathrm{curl}\,\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\mathrm{div}\,\mathbf{u}\|^2_{L^2(\Omega)}. \end{aligned}$$

Note that $X_{\mathrm{T}}(\Omega)$ is a Hilbert space with an inner product defined in [11]. As previously shown, any non-trivial eigenfunction of (2.4) is tangential. Since $X_{\mathrm{T}}(\Omega)$ only consists of tangential fields, this supports our choice of studying (1.4) instead of other modifications of the original Steklov-Maxwell Problem.

Recall that the strong formulation of the modified Steklov-Maxwell problem, as stated in [11], is to find $\mathbf{u} \in X_{\mathrm{T}}(\Omega)$, $\lambda \in \mathbb{C}$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} - \theta \operatorname{grad}\operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \nu \times (\operatorname{curl}\mathbf{u}) = \lambda \mathbf{u} & \text{on } \Gamma. \end{cases} \tag{2.8}$$

Here, $\alpha \in \mathbb{R}$ and $\theta > 0$ are parameters, and $\operatorname{grad}\operatorname{div}\mathbf{u} = \nabla(\nabla \cdot \mathbf{u})$ is the gradient of the divergence of $\mathbf{u}$. The corresponding weak problem is to find constants $\lambda \in \mathbb{C}$ and vector fields $\mathbf{u} \in X_{\mathrm{T}}(\Omega)$ such that for each test function $\mathbf{v} \in X_{\mathrm{T}}(\Omega)$,

$$\int_{\Omega} \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v}\,\mathrm{d}x - \alpha \int_{\Omega} \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x + \theta \int_{\Omega} \operatorname{div}\mathbf{u}\ \operatorname{div}\mathbf{v}\,\mathrm{d}x = -\lambda \int_{\Gamma} \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}s. \tag{2.9}$$

As in the Steklov-Laplace case, the weak formulation is equivalent to the strong formulation, provided that the weak eigenfunctions are sufficiently regular. Furthermore, Lamberti & Stratis prove in [11] that the eigenvalues of the weak problem form a monotonically decreasing sequence of real numbers tending to negative infinity. In particular, they solve the issue in [3] where the original Steklov-Maxwell spectrum consists of two sequences, one increasing to infinity and the other tending to zero.

Note that (2.9) resembles the weak form of the Steklov-Laplace problem (2.2), except for the terms with the parameters α and θ. For instance, writing the Laplacian of a function u as $\Delta u = \nabla \cdot \nabla u$, we see that the terms

$$\int_{\Omega} \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v}\,\mathrm{d}x, \ \int_{\Omega} \nabla u \cdot \nabla v\,\mathrm{d}x$$

both involve the first-order differential operator used to define the curl-curl and Laplace operators. Additionally, the right-hand side of both equations involves an integral over the boundary and the multiplication of an eigenfunction and a test function. The likeness of the two problems demonstrates that the modified Steklov-Maxwell problem is a vectorial analog of the Steklov-Laplace problem. In Section 2.6, we see further common features of the two problems in their variational formulations and homothety properties.

2.5.2 Strong Implies Weak

This section corresponds to the logical implication between the strong formulation and weak formulation in Figure 1.1. We show that any solution to (2.8) is a solution to (2.9). First, we need two integration by parts formulas, adapted from [11].

Lemma 1. *Let* $\mathbf{E}$ *and* v *be sufficiently regular vector and scalar fields, respectively. Then*

$$\int_{\Omega} v\ \operatorname{div}\mathbf{E}\,\mathrm{d}x = \int_{\Gamma} v\mathbf{E} \cdot \mathbf{n}\,\mathrm{d}s - \int_{\Omega} \operatorname{grad} v \cdot \mathbf{E}\,\mathrm{d}x.$$

Lemma 2. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be sufficiently regular vector fields. Then*

$$\int_\Omega \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v}\,\mathrm{d}x = \int_\Omega \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x - \int_\Gamma (\nu\times\operatorname{curl}\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}s.$$

Lemmas 1 and 2 follow by integrating the vector identities

$$\operatorname{div}(v\mathbf{E}) = v\operatorname{div}\mathbf{E} + \operatorname{grad}v\cdot\mathbf{E}, \tag{2.10}$$

$$\operatorname{div}\mathbf{u}\times\mathbf{v} = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}, \tag{2.11}$$

and applying the Divergence Theorem. We now have the following result:

Theorem 5. *Any solution of (2.8) is a solution of (2.9).*

Proof. Suppose $(\lambda,\mathbf{u})\in\mathbb{C}\times X_{\mathrm{T}}(\Omega)$ is an eigenpair of (2.8). Consider any $\mathbf{v}\in X_{\mathrm{T}}(\Omega)$. Take the dot product on both sides of the equation in (2.8) and integrate over Ω to get

$$\int_\Omega \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x - \alpha\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x - \theta\int_\Omega (\operatorname{grad}\operatorname{div}\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x = 0. \tag{2.12}$$

Applying Lemma 1 to the last term of (2.12), we have

$$\int_\Omega (\operatorname{grad}\operatorname{div}\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x = \int_\Gamma (\operatorname{div}\mathbf{u})\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s - \int_\Omega \operatorname{div}\mathbf{u}\ \operatorname{div}\mathbf{v}\,\mathrm{d}x.$$

But $\mathbf{v}\in X_{\mathrm{T}}(\Omega)$ means that $\mathbf{v}$ is tangential, so the surface integral vanishes.

Now, apply Lemma 2 to the first integral of (2.12) to get

$$\int_\Omega \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v}\,\mathrm{d}x + \int_\Gamma (\nu\times\operatorname{curl}\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}s = \int_\Omega \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v}\,\mathrm{d}x + \lambda\int_\Gamma \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}s.$$

Simplifying returns (2.9). □

2.5.3 Weak Implies Strong, with Assumptions

We can also show that any solution to (2.9) is a solution to (2.8), assuming the solution is sufficiently regular.

Theorem 6. *A sufficiently regular solution of (2.9) is also a solution of (2.8).*

Proof. Suppose $\mathbf{u}\in X_{\mathrm{T}}(\Omega)$ is an eigenfunction of the weak problem with corresponding eigenvalue $\lambda\in\mathbb{C}$. Assume $\mathbf{u}$ is sufficiently regular, so that we may take its second order partial derivatives. Then reversing the steps in Theorem 5, we arrive at the equation

$$\int_\Omega (\operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} - \theta\operatorname{grad}\operatorname{div}\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x = \int_\Gamma (\nu\times\operatorname{curl}\mathbf{u} - \lambda\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}s$$

for every $v \in X_{\mathrm{T}}(\Omega)$. Note that the left-hand side depends only on Ω, an open set, and hence we may assume $\mathbf{v} = 0$ on Γ. Thus,

$$\int_\Omega (\operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} - \theta \operatorname{grad}\operatorname{div}\mathbf{u}) \cdot \mathbf{v}\, \mathrm{d}x = 0$$

in Ω. Since this holds for any test function $\mathbf{v}$, we conclude that

$$\operatorname{curl}\operatorname{curl}\mathbf{u} - \alpha\mathbf{u} - \theta \operatorname{grad}\operatorname{div}\mathbf{u} = 0$$

in Ω. Similarly, on Γ we have

$$\nu \times (\operatorname{curl}\mathbf{u}) - \lambda\mathbf{u} = 0.$$

Therefore, both equations of (2.8) are satisfied. □

2.6 Properties of the Modified Steklov-Maxwell Problem

Given the weak formulation (2.9), we can generalize some properties of the Steklov-Laplace problem. We will see that additional assumptions are needed to generalize the latter's properties.

Let $A_1, A_2, \dots$ denote the eigenvalues of the problem

$$\int_\Omega \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v}\, \mathrm{d}x + \theta \int_\Omega \operatorname{div}\mathbf{u}\, \operatorname{div}\mathbf{v}\, \mathrm{d}x = A \int_\Omega \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}x,\ \forall \mathbf{v} \in (H_0^1(\Omega))^3,$$

where $H_0^1(\Omega)$ is the subspace of the first Sobolev space $H^1(\Omega)$ consisting of functions that vanish on Γ. The following theorem from [11] holds.

Theorem 7 (The Variational Formulation of the Modified Steklov-Maxwell Problem)**.** *Suppose $\alpha \neq A_k$, for each $k \in \mathbb{N}$. Then the n-th eigenvalue of (2.9) is*

$$\lambda_n = - \min_{\substack{V \subseteq X_{\mathrm{T}}(\Omega) \\ \dim V = n}} \max_{\mathbf{v} \in V \setminus (H_0^1(\Omega))^3} \frac{\int_\Omega (|\operatorname{curl}\mathbf{v}|^2 - \alpha|\mathbf{v}|^2 + \theta|\operatorname{div}\mathbf{v}|^2)\, \mathrm{d}x}{\int_\Gamma |\mathbf{v}|^2\, \mathrm{d}s}. \tag{2.13}$$

Furthermore, each eigenvalue is of finite multiplicity.

Thus, there is a well-defined variational formulation of the modified Steklov-Maxwell problem, assuming α is carefully chosen. Moreover, (2.13) is equivalent to (2.9). We require $\mathbf{v} \in V \setminus (H_0^1(\Omega))^3$ so that the proof Lamberti & Stratis [11] present is valid. They characterize the λ_k using the eigenvalues of a related linear operator whose eigenvalues correspond to those of (2.8). To avoid an infinite-dimensional eigenspace, they insist that $\mathbf{v} \notin (H_0^1(\Omega))^3$. Additionally, the condition that α does not agree with any of the A_k ensures that the weak formulation (2.9) does not reduce to a different eigenvalue problem.

We conclude by stating and proving an original result generalizing Theorem 4.

Theorem 8 (Homothety Property for the modified Steklov-Maxwell Eigenproblem)**.**
Let $\alpha \in \mathbb{R}$*,* $\theta > 0$*. Suppose* $\alpha \neq A_k$ *for all* $k \in \mathbb{N}$*. Let* $c > 0$ *be given and denote by* Ω_c *a homothety of* Ω *with ratio* c*. Then for every* $n \in \mathbb{N}$*,*

$$\lambda_n(\alpha, \theta, \Omega) = c\lambda_n(\frac{\alpha}{c^2}, \theta, \Omega_c). \tag{2.14}$$

Proof. Suppose $(\lambda, \mathbf{u}) \in \mathbb{R} \times X_{\mathrm{T}}(\Omega)$ is an eigenpair of (1.2) for fixed parameters α and θ. Then for each $\mathbf{v} \in X_{\mathrm{T}}(\Omega)$,

$$\int_{\Omega} \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v} - \alpha\mathbf{u} \cdot \mathbf{v} + \theta \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} \, \mathrm{d}x = -\lambda \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}s. \tag{2.15}$$

Consider the change of variables $x_i = y_i/c$, for $i = 1, 2, 3$. Under this change of variables, linear dimensions are scaled by $1/c$, so areas and volumes are scaled by $1/c^2$ and $1/c^3$, respectively. Furthermore,

$$\frac{\partial g}{\partial x_j} = \frac{\partial g}{\partial y_j}\frac{\partial y_j}{\partial x_j} = c\frac{\partial g}{\partial y_j}$$

for all differentiable scalar fields g, so the curl and div operators are each scaled by c. Thus, (2.15) becomes

$$\int_{\Omega_c} c^2(\operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v} - \frac{\alpha}{c^2}\mathbf{u} \cdot \mathbf{v} + \theta \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v})\frac{1}{c^3} \, \mathrm{d}y = -\lambda \int_{\partial\Omega_c} \mathbf{u} \cdot \mathbf{v} \, \frac{1}{c^2} \, \mathrm{d}s.$$

Simplifying gives that λ/c is an eigenvalue for (2.9) with parameters (α/c^2, θ) on the domain Ω_c. Applying Theorem 7 gives (2.14). □

Theorem 8 illustrates that knowing the eigenvalues for a given domain and parameters provides the eigenvalues for infinitely many other problems. Note that this result is not a direct generalization of Theorem 4, as we must scale the parameters involved in the modified Steklov-Maxwell problem. However, if $\alpha = 0$, the eigenvalues are scaled in relation only to the homothety of Ω (the parameters of the two problems are the same). In this case, we have a natural generalization of Theorem 4.

In this chapter, we have examined the common features of eigenvalue problems through the Steklov-Laplace and modified Steklov-Maxwell problems. We have stated the various formulations of the latter and discussed the different motivations for studying it. With the theoretical foundation of the problem explained, we now begin our study of its numerical framework.

Chapter 3

The Discrete Modified Steklov-Maxwell Eigenproblem

This chapter contains the core material of this work. We first introduce the finite element method. We then present our main results characterizing the discrete formulation of the modified Steklov-Maxwell eigenvalue problem and discuss its convergence and conditioning.

3.1 Finite Element Method

In this section, we follow the notation, definitions, and theorems of Braess [2].

3.1.1 The Characterization Theorem

Let V be a real vector space. We call a function $a\colon V \times V \to \mathbb{R}$ a *bilinear form* if $a(\cdot,\cdot)$ is linear in each of its arguments. We say $a(\cdot,\cdot)$ is *symmetric* if $a(x,y) = a(y,x)$ for all $x, y \in V$. Also, $a(\cdot,\cdot)$ is *positive* if $a(x,x) > 0$ for each non-zero $x \in V$. A linear transformation $l\colon V \to \mathbb{R}$ is called a *linear functional*.

One of the most important theorems in finite element theory is the Characterization Theorem, as stated in [2].

Theorem 9 (Characterization Theorem)**.** *Suppose V is a vector space over $\mathbb{R}$ and let $a\colon V \times V \to \mathbb{R}$ be a symmetric positive bilinear form. Additionally, let l be a linear functional on V. Then the function*

$$J(v) := \frac{1}{2}a(v,v) - l(v)$$

attains its minimum over V at u if and only if

$$a(u,v) = l(v) \tag{3.1}$$

for every $v \in V$. Furthermore, if (3.1) has a solution, it is unique.

The Characterization Theorem gives us an equivalence between the variational form and the weak form of a problem. In general, solutions may not exist. However, the work of Lamberti

& Stratis in [11] and Ferraresso, Lamberti & Stratis in [4] settle this issue; we may assume there is a solution to the modified Steklov-Maxwell problem.

The Characterization Theorem as stated above does not explicitly apply to eigenvalue problems. In such problems, we have two symmetric bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. We are looking for constants λ and $u \in V$ such that $a(u,v) = \lambda b(u,v)$ for every $v \in V$. While the Characterization Theorem does not apply to this problem, the concepts it illustrates are useful for our purposes. Boffi [1] describes the abstract characterizations of eigenvalue problems using bilinear forms. The theory is illustrated in Section 2.2 in Theorems 1, 2, and 3. Namely, Boffi's paper shows that the weak and variational formulations of general eigenvalue problems are equivalent, provided we put certain assumptions on the bilinear forms. Furthermore, we can formulate an abstract min-max characterization of the eigenvalues. The proofs are similar to those presented in Section 2.2, so our previous analysis of the Steklov-Laplace problem hints at a general theory for studying eigenvalue problems.

3.1.2 Finite Elements

Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be symmetric bilinear forms on a vector space V (typically a function space) and assume a is coercive (see [2]). The problem of finding $u \in V$ and $\lambda \in \mathbb{C}$ such that

$$a(u,v) = \lambda b(u,v) \tag{3.2}$$

for all $v \in V$ is a daunting task, as V is typically infinite-dimensional. However, working with finite-dimensional spaces is significantly easier. Thus, we approximate solutions of (3.2) by looking instead for solutions in some finite-dimensional space V_h. The finite element method involves carefully choosing the space V_h.

A *finite element* is a triplet $(K, P_n(K), \Sigma(K))$, where K is a geometric domain (usually a triangle in 2D or tetrahedron in 3D), $P_n(K)$ is the space of (possibly multivariate) polynomials of degree at most n on K, and $\Sigma(K)$ is the space of linear functionals dual to $P_n(K)$. Finite elements are used to build a *finite element space* that we can use to discretize a given variational formulation.

Suppose $\Omega_h \subseteq \mathbb{R}^3$ is a bounded polygonal domain. We assume Ω_h is polygonal so that we can approximate it exactly by tetrahedra. Let $h > 0$ and suppose $\mathcal{T}_h$ is some partition of Ω_h into finitely many tetrahedra, each with edge lengths at most $2h$. We call $\mathcal{T}_h$ a *triangulation* of Ω_h. We say $\mathcal{T}_h$ is *admissible* if

1. $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$;
2. For each $T, T' \in \mathcal{T}_h$, if $T \cap T'$ consists of exactly one point, then the point is a shared vertex of T and T';
3. For each $T, T' \in \mathcal{T}_h$, if $T \cap T'$ consists of more than one point, then $T \cap T'$ is a shared face of T and T'.

Assume now that $\Omega \subseteq \mathbb{R}^3$ is a bounded (not necessarily polygonal) domain with boundary $\Gamma = \partial\Omega$ of class $C^{1,1}$. Let $\mathcal{T}_h$ be a triangulation of Ω. We define the *mesh* to be $\Omega_h \coloneqq \bigcup_{T \in \mathcal{T}_h} T$ and assume $\mathcal{T}_h$ is admissible. The quantity $h > 0$ is the *mesh size*, signifying the maximum edge length among the tetrahedra. The domain Ω_h is essentially a polygonal domain approximating Ω. As h tends to zero, Ω_h becomes a better approximation of Ω (see Figure 3.1). On each $T \in \mathcal{T}_h$, there is a corresponding finite element $(T, P_n(T), \Sigma(T))$. We let V_h consist of all vector fields $\mathbf{u}_h = (u_h^1, u_h^2, u_h^3)$ such that $u_h^i|_T \in P_n(T)$ for all $T \in \mathcal{T}_h$, and u_h^i is piecewise continuous on Ω_h for $i = 1, 2, 3$. The space V_h is called the *finite element space*. Since we assume that u_h^i is continuous over the edges of Ω_h, the values of u_h^i must agree on each vertex of the shared edge. For instance, if e is a shared edge of tetrahedra $T_1, T_2 \in \mathcal{T}_h$ and v_1, v_2 are the vertices of e, then $\mathbf{u}|_{T_1}(v_1) = \mathbf{u}|_{T_2}(v_1)$ and $\mathbf{u}|_{T_1}(v_2) = \mathbf{u}|_{T_2}(v_2)$. Note there are numerous other possible continuity assumptions, but we work only with piecewise continuous finite elements in this thesis. The union of the standard basis of each $P_n(T)$ forms a basis for V_h, as the function values in each tetrahedron are well-defined by $\Sigma(T)$. Since each $P_n(T)$ is finite-dimensional, the finite element space is finite-dimensional. Solutions to (3.2) are now approximated by solutions to the *discrete problem* of finding $\mathbf{u}_h \in V_h$ and $\lambda_h \in \mathbb{C}$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = \lambda_h b(\mathbf{u}_h, \mathbf{v}_h) \tag{3.3}$$

for every $\mathbf{v}_h \in V_h$. This is the finite element method for vectorial problems; a similar approach is used for scalar problems.

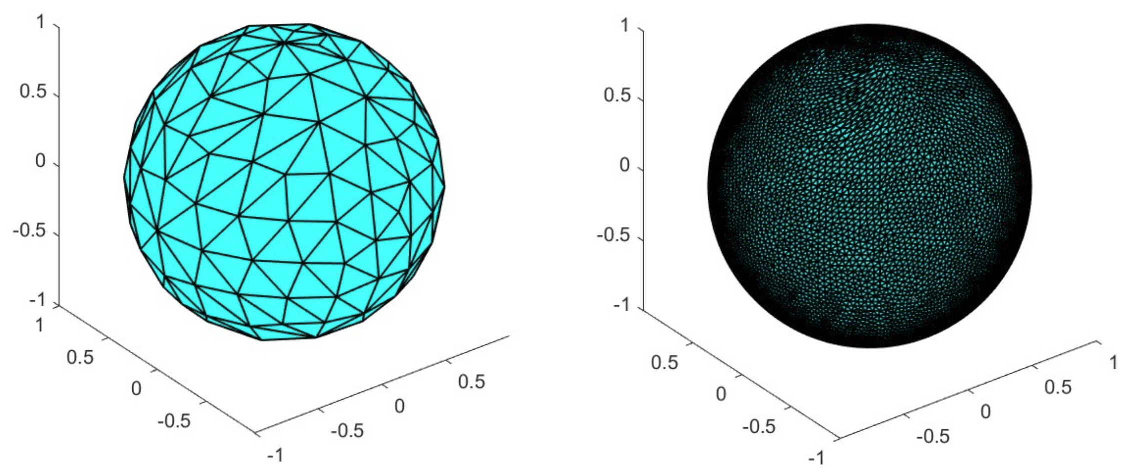


Figure 3.1: Two meshes approximating the unit sphere (built in FreeFem++). The right image illustrates how taking h to be small results in a better approximation of the sphere.

3.1.3 A Two-Dimensional Example

Although we defined finite elements for three-dimensional domains, everything described above holds for two-dimensional domains. We consider a simple two-dimensional example to illustrate the above concepts. In the following, we only consider scalar functions.

Consider the triangle in Figure 3.2 defined by the vertices x_1, x_2, x_4, which we denote by $\triangle x_1x_2x_4$. The interior of $\triangle x_1x_2x_4$ is a domain, so we can create a mesh by adding an edge x_2x_3, which partitions the domain into triangles $T_1 = \triangle x_1x_2x_3$ and $T_2 = \triangle x_2x_3x_4$. For $i = 1, 2$, we define the finite elements $(T_i, P_1(T_i), \Sigma(T_i))$. Note that the polynomials in $P_1(T_i)$ are in two variables, so spaces $P_1(T_i)$ and $\Sigma(T_i)$ are three-dimensional over $\mathbb{R}$. The finite element space, which we denote by V, consists of functions u such that $u|_{T_i}$ is a linear polynomial for each $i = 1, 2$. As we are assuming u is continuous over the edge x_2x_3, u must satisfy $u|_{T_1}(x_2) = u|_{T_2}(x_2)$ and $u|_{T_1}(x_3) = u|_{T_2}(x_3)$.

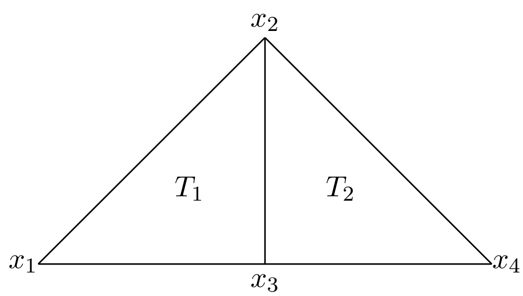


Figure 3.2: An example mesh in two dimensions.

To demonstrate the relevance of the dual spaces $\Sigma(T_i)$ in the definition of finite elements, we show that u is completely determined by its values on the vertices. Without loss of generality, we show this fact for T_1. Consider the basis of $P_1(T_1)$, $\mathcal{B}_1 = \{p_1, p_2, p_3\}$, defined by $p_i(x_j) = 1$ if $i = j$ and 0 otherwise. Corresponding to this basis is a basis of $\Sigma(T_1)$, defined as $\mathcal{B}_1^* = \{\phi_1, \phi_2, \phi_3\}$, such that $\phi_i(p_j) = 1$ if $i = j$ and 0 otherwise. We say $\mathcal{B}_1^*$ is the basis dual to $\mathcal{B}_1$, as $\Sigma(T_1)$ is the dual space of $P_1(T_1)$. Recall that $\Sigma(T_1)$ consists of linear functionals, so the ϕ_i are linear transformations from $P_1(T_1)$ to $\mathbb{R}$. Given a function $u \in V$, we can write $v = u|_{T_1}$ as $v = \phi_1(v)p_1 + \phi_2(v)p_2 + \phi_3(v)p_3$. By the definition of the p_i, we have that

$$v(x_j) = \sum_{i=1}^{3} \phi_i(v)p_i(x_j) = \phi_j(v)p_j(x_j) = \phi_j(v).$$

That is, defining v on the vertices of T_1 completely determines v in and on T_1. The same argument applies for $u|_{T_2}$ with the vertices x_2, x_3, x_4.

Given this result, we can link the bases of $\Sigma(T_1)$ and $\Sigma(T_2)$. Suppose $\mathcal{B}_2 = \{q_2, q_3, q_4\}$ is the basis of $P_1(T_2)$ satisfying $q_i(x_j) = 1$ if $i = j$ and 0 otherwise. Let $\mathcal{B}_2^* = \{\psi_2, \psi_3, \psi_4\}$

be the basis of $\Sigma(T_2)$ dual to $\mathcal{B}_2$ (in the same way that $\mathcal{B}_1^*$ is dual to $\mathcal{B}_1$). Then

$$\phi_2(u) = u|_{T_1}(x_2) = u|_{T_2}(x_2) = \psi_2(u),$$

and similarly $\phi_3(u) = \psi_3(u)$. As u is an arbitrary function in V, these equalities hold for every $u \in V$. Hence, the continuity assumptions we make connect the bases $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$.

3.1.4 Matrices and Approximation

The power of V_h being finite-dimensional is that we can now work with matrices. Suppose $\mathcal{B} = \{\boldsymbol{\varphi_1}, \ldots, \boldsymbol{\varphi_N}\}$ is a basis of V_h. We can define matrices A and B by

$$(A)_{ij} = a(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j),\ (B)_{ij} = b(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j)$$

for all $(i, j) \in \{1, \ldots, N\}^2$, where $(A)_{ij}$ is the entry in the i-th row and j-th column of A (and similarly for B). Since $\mathcal{B}$ is a basis of V_h, our solution $\mathbf{u}_h$ can be written as a linear combination of the basis elements. That is, $\mathbf{u}_h = c_1\boldsymbol{\varphi_1} + \ldots + c_N\boldsymbol{\varphi_N}$ for some $c_1, \ldots, c_N \in \mathbb{R}$. Thus, choose $c = (c_1, \ldots, c_N)^T$ to be the coordinate vector of $\mathbf{u}_h$, so that $Ac = \lambda_h Bc$. Hence, solving (3.3) for $\mathbf{u}_h$ amounts to solving this *generalized eigenvalue problem* for the vector c. Using existing algorithms in FreeFem++ and MATLAB, we can construct these matrices and solve this generalized eigenvalue problem, given the bilinear forms a and b. We briefly describe the algorithms used in FreeFem++ (in the `EigenValue` function) and MATLAB (in the `eigs` function) to compute eigenvalues in Section 3.2.

3.1.5 The Discrete Steklov-Maxwell Problem

Given the theoretical framework of discretizing eigenvalue problems described in the previous section, we can now set up a discrete formulation of the modified Steklov-Maxwell problem. In doing so, we transition from the weak problem (a continuous problem) to a problem that computers can interpret. This section corresponds to the arrow from the weak formulation box to the discrete formulation box in Figure 1.1.

From Section 2.5, the weak form of the modified Steklov-Maxwell problem is to find $\mathbf{u} \in X_{\mathrm{T}}(\Omega)$, $\lambda \in \mathbb{C}$ so that for all $v \in X_{\mathrm{T}}(\Omega)$,

$$\int_{\Omega} \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v} - \alpha\mathbf{u}\cdot\mathbf{v} + \theta\operatorname{div}\mathbf{u}\ \operatorname{div}\mathbf{v}\,\mathrm{d}x = -\lambda \int_{\Gamma} \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}s, \tag{3.4}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with sufficiently regular boundary $\Gamma = \partial\Omega$. Notice that both sides of the equation correspond to bilinear forms, and the grad-div term in the strong formulation (2.8) ensures that the left-hand side is coercive (see [11]). Let $\mathcal{T}_h$ be a triangulation of Ω, and let Ω_h be the corresponding mesh with boundary $\partial\Omega_h$. The *discrete* *modified Steklov-Maxwell problem* is to find $\mathbf{u}_h \in V_h$ and $\lambda_h \in \mathbb{C}$ such that for any $\mathbf{v}_h \in V_h$,

$$\int\limits_{\Omega_h} \operatorname{curl} \mathbf{u}_h \cdot \operatorname{curl} \mathbf{v}_h - \alpha \mathbf{u}_h \cdot \mathbf{v}_h + \theta \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h \,\mathrm{d}x = -\lambda_h \int\limits_{\partial\Omega_h} \mathbf{u}_h \cdot \mathbf{v}_h \,\mathrm{d}s. \tag{3.5}$$

As in Section 3.1.4, we can set up a linear system and approximate the modified Steklov-Eigenvalues, using the C++ mathematical library (FreeFem++) and MATLAB.

Let us explicitly describe the finite element matrices. Suppose $\mathcal{B} = \{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_N\}$ is the finite element basis of V_h. We define four matrices, A_1, A_2, A_3, and B, corresponding to the four terms appearing in (3.5). The matrices are given by

$$(A_1)_{ij} = \int\limits_{\Omega_h} \operatorname{curl} \boldsymbol{\varphi}_i \cdot \operatorname{curl} \boldsymbol{\varphi}_j \,\mathrm{d}x, \; (A_2)_{ij} = \int\limits_{\Omega_h} \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \,\mathrm{d}x,$$
$$(A_3)_{ij} = \int\limits_{\Omega_h} \operatorname{div} \boldsymbol{\varphi}_i \operatorname{div} \boldsymbol{\varphi}_j \,\mathrm{d}x, \; (B)_{ij} = \int\limits_{\Gamma_h} \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \,\mathrm{d}s$$

for every $(i, j) \in \{1, \dots, N\}^2$. Each of these matrices is symmetric by the commutativity of the dot product. Furthermore, the matrices are all real and *sparse* (have relatively few non-zero entries), stemming from the finite element method. Letting $A = A_1 - \alpha A_2 + \theta A_3$, we approximate the eigenvalues of (3.4) by solving the generalized eigenvalue problem $Ac = -\lambda_h Bc$ for $\lambda_h \in \mathbb{C}$ and $c \in \mathbb{C}^N$. Note that the vector c is the coordinate vector of our approximate eigenfunction with respect to the basis $\mathcal{B}$.

We let h denote the *mesh size* of Ω_h, quantifying the maximum size of the edges making up the approximate domain. As h approaches zero, our partition of Ω becomes increasingly finer, and Ω_h ultimately becomes Ω. For our approximations to be useful, we need evidence that λ_h approaches a true eigenvalue λ as h vanishes. If $\lim_{h\to 0} \lambda_h = \lambda$, we say λ_h *converges* to λ as h tends to zero. Section 3.3 illustrates convergence for the Steklov-Laplace problem and the modified Steklov-Maxwell problem.

We conclude this section with a comment on the space V_h. We say that our method is *conforming* if V_h is a subspace of $X_{\mathrm{T}}(\Omega)$, otherwise it is *nonconforming* [2]. In our case, we must explicitly tell FreeFem++ that $\mathbf{u}_h \cdot \nu = 0$ on Γ to get a conforming method. The finite element matrices (the A_i and B) are drastically different if we choose conforming finite elements rather than nonconforming. The main contribution of this work is showing that conforming methods are critical for developing the numerical framework of the modified Steklov-Maxwell problem.

3.2 Numerically Determining the Eigenvalues

This section is adapted from Trefethen & Bau [16].

3.2.1 Shift-Invert Method

Before discussing the main results of this thesis, we briefly describe methods to compute the eigenvalues of the finite element matrices. This section completes the theory behind approximating solutions to eigenvalue problems with the finite element method.

Let $A, B \in \mathbb{C}^{n\times n}$. A *generalized eigenvalue problem* is to find non-zero $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ such that $Ax = \lambda Bx$. Rearranging, this becomes $(A - \lambda B)x = 0$. Since x must be non-zero, $A - \lambda B$ is singular, so determining λ amounts to finding the roots of the polynomial $p(\lambda) = \det(A - B\lambda)$. However, exact root finding is no simple task as solvability of the roots in terms of radicals is not guaranteed when $\deg(p) \geq 5$. We instead use iterative methods involving matrices to determine the eigenvalues numerically.

Both the `EigenValue` function in FreeFem++ and `eigs` function in MATLAB make use of the ARPACK software [13]. These functions input two matrices and return the eigenvalues and eigenvectors. For example, we may input $A_1 - \alpha A_2 + \theta A_3$ and B, and the functions output the solutions to the generalized eigenvalue problem

$$(A_1 - \alpha A_2 + \theta A_3)c = -\lambda Bc.$$

In MATLAB, these are returned in two separate matrices, whereas in FreeFem++ they are organized into a list.

Since the finite element matrices are sparse, singular, and symmetric, the algorithms use the shift-invert method in ARPACK. We define a shift $\sigma \in \mathbb{R}$ and compute the eigenvalues near σ by solving the eigenvalue problem

$$(A - \sigma B)^{-1}Bx = \nu x, \tag{3.6}$$

where $\nu = 1/(\lambda - \sigma)$. Once we know σ and ν, we can simply solve for λ. Section 4.5 of [13] details this method extensively. With this setup, we must now compute the eigenvalues.

We compute the eigenvalues with iterative methods, as non-iterative approaches often require an immense number of operations. Numerous iterative methods may be employed depending on the problem at hand. Iterative methods are particularly useful for problems involving sparse matrices, as significantly fewer operations are required to compute the eigenvalues. Furthermore, iterative solvers are especially good for computing larger magnitude eigenvalues, so the shift-invert method above provides a link to determining small-magnitude eigenvalues with high accuracy.

3.2.2 Power Iteration

The power method is one of the simplest iterative methods to determine eigenvalues. Suppose $A \in \mathbb{R}^{n\times n}$ has real eigenvalues $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$. Let $\{x_1, \ldots, x_n\}$ be a set of orthonormal eigenvectors of A such that $Ax_i = \lambda_i x_i$ for all $i = 1, \ldots, n$. Suppose $v_0 \in \mathbb{R}^n$

is an arbitrary vector of unit norm. The power method simply iterates using powers: at the k-th step, we set $v_k = Av_{k-1}/\|Av_{k-1}\|$, where $k \in \mathbb{N}$ and $\|\cdot\|$ is the Euclidean norm. The claim is that v_k converges to x_1. To see this, expand v_0 in terms of the eigenvectors as

$$v_0 = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

for some $a_1, \ldots, a_n \in \mathbb{R}$. Now, multiply by A^k on both sides of the above equation to get

$$\begin{aligned} v_k = c_kA^kv_0 &= c_k(a_1A^kx_1 + \ldots + a_nA^kx_n) \\ &= c_k(a_1\lambda_1^kx_1 + \ldots + a_n\lambda_n^kx_n) \\ &= c_k\lambda_1^k\left(a_1x_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \ldots + a_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x_n\right) \end{aligned}$$

for some constant $c_k > 0$. Since $\lambda_1 > \lambda_j$ for each $j = 1, \ldots, n$, as k grows we isolate the eigenvector corresponding to the eigenvalue of largest magnitude, assumed to be λ_1 in the above. That is, the first eigenvalue dominates the other eigenvalues. This is the power method.

The power method is a simple but ineffective way to determine the eigenvalues in practice. For instance, it can isolate only the eigenvector corresponding to the eigenvalue of the largest magnitude, which is very dependent on the other eigenvalues. Numerically, if the two largest eigenvalues are very close, then the power method is insufficient as the algorithm becomes very slow: one eigenvalue must be distinctly larger than the others. We instead use the Arnoldi process that FreeFem++ and MATLAB are based on.

3.2.3 Arnoldi Iteration

We begin with some basic definitions. We call a matrix $M \in \mathbb{R}^{n\times n}$ *unitary* if

$$MM^* = M^*M = I,$$

where I denotes the identity matrix and M^* the conjugate transpose of M. We say M is *upper Hessenberg* if $(M)_{ij} = 0$ whenever $i > j + 1$. That is, all the entries of M below the first sub-diagonal are zero.

For the Arnoldi process, we assume $A \neq A^*$ and that there are $n \times n$ matrices U and H, where U is unitary and H is upper Hessenberg, such that $A = UHU^*$. Let $u_1, \ldots, u_n$ denote the columns of U and let $\tilde{U}_m = [u_1|u_2|\ldots|u_m]$ for some fixed $1 \leq m < n$. Furthermore, let $\tilde{H}_m$ denote the upper left $(m+1) \times m$ block of H. Then $A\hat{U}_m = \hat{U}_{m+1}\hat{H}_m$. In particular, if $h_{ij} = (H)_{ij}$, then

$$Au_m = \sum_{k=1}^{m+1} h_{km}q_k,$$

so we get a recurrence relation for q_{m+1}.

The Arnoldi iteration naturally follows from the above recurrence relation. We begin with some arbitrary unit vector $u_1 \in \mathbb{R}^n$. We iterate through the integers $k \geq 2$. First set $v = Au_k$, where $k \geq 2$. Now for each $l \in \mathbb{N}$ satisfying $1 \leq l \leq k$, set $h_{jk} = u_j^* v$, and let $v' = v - h_{jk}$. Finally, take $h_{k+1,k} = \|v'\|$ and $u_{k+1} = v'/h_{k+1,k}$. The result is a sequence of orthonormal vectors $u_1, u_2, \ldots, u_m$ and an upper Hessenberg matrix H_m whose ij-th entry is h_{ij}. Furthermore, $H_m = U_m^* A U_m$, where the l-th column of U_m is u_l, for $l = 1, \ldots, m$. It turns out that the eigenvalues of H_m are excellent approximations to the eigenvalues of A. The simple structure of H_m means that its eigenvalues can be computed efficiently. We refer the reader to Trefethen & Bau [16] for an in-depth study of these algorithms. Overall, the Arnoldi method is more reliable than the power method as it does not depend on one eigenvalue dominating the others.

3.2.4 Condition Numbers

Each of the iterative algorithms we use accumulates errors at each step, as computers use floating-point arithmetic. The idea of convergence described in Section 3.1.5 is essential, but we should also characterize how sensitive linear systems are to small perturbations. This is the idea of *conditioning*.

Consider the linear equation $Ax = b$, where $A \in \mathbb{C}^{n \times n}$ and $x, b \in \mathbb{C}^n$. We wish to characterize how sensitive x is to small changes in b and vice versa. The *condition number* of a matrix is one way of characterizing this. We define the condition number of A as $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$, where $\kappa(A) = \infty$ if A is not invertible. Here, $\|A\|_2$ is the matrix 2-norm, defined by

$$\|A\|_2 = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|},$$

where $\|\cdot\|$ is the Euclidean norm for vectors. The matrix A is called *well-conditioned* if $\kappa(A)$ is small, and *ill-conditioned* otherwise. A small condition number corresponds to small perturbations to x or b having little impact on the solutions to the equation $Ax = b$. Likewise, a large condition number signifies that small changes lead to considerable changes in the solutions. In practice, the condition number can be approximated by the quotient of the largest eigenvalue of A with the smallest eigenvalue. See [16] for a deeper analysis.

The power and Arnoldi iterations both involve matrix multiplication with A, so determining the condition numbers is critical for our analysis. However, we have not stated a precise definition of what it means for a condition number to be "small" or "large," as this depends on the specific problem. We illustrate this idea in the next section by comparing condition numbers for matrices built with two different procedures.

3.3 Numerical Results

In this section, all computations are done on the unit disk in $\mathbb{R}^2$ for the Steklov-Laplace problem and the unit cube $(0,1)^3$ in $\mathbb{R}^3$ for the modified Steklov-Maxwell problem.

3.3.1 Results for the Steklov-Laplace Problem

We begin this section with a brief discussion of the convergence of the finite element method for the Steklov-Laplace problem. Recall, the weak formulation of this problem is to find $u \in H^1(\Omega)$ and $\sigma \in \mathbb{C}$ such that

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, \mathrm{d}x = \sigma \int_\Gamma uv \, \mathrm{d}s \tag{3.7}$$

for all $v \in H^1(\Omega)$. Thus, the discrete formulation involves two matrices, corresponding to each term of the weak problem. The key question is whether the eigenvalues computed with the finite element matrices converge to the true eigenvalues.

Although we described the finite element method for vectorial problems, the finite element approximation of scalar functions is identical. Let the mesh size $h > 0$ be given. Suppose $V_h \subseteq H^1(\Omega)$ is the finite element space and Ω_h the mesh with boundary Γ_h. Let $\{\phi_1, \ldots \phi_m\}$ be the standard basis of V_h. We define matrices A and B by

$$(A)_{ij} = \int_{\Omega_h} \operatorname{grad} \phi_i \cdot \operatorname{grad} \phi_j \, \mathrm{d}x, \; (B)_{ij} = \int_{\Gamma_h} \phi_i \; \phi_j \, \mathrm{d}s.$$

Approximating the Steklov-Laplace eigenvalues/eigenvectors thus amounts to solving for the eigenvalues $\sigma \in \mathbb{C}$ and eigenvectors $c \in \mathbb{C}^m$ of the linear equation $Ac = \sigma Bc$.

For the unit disk, let $\sigma^{(k)}$ denote the true k-th eigenvalue of (3.7) and $\sigma_h^{(k)}$ the approximation to it. By [5], we know that the true eigenvalues on the unit disk are $\sigma_n = n$ for all non-negative integers n. Furthermore, each eigenvalue is of multiplicity two, except for σ_0 which occurs once. According to Boffi [1], the error should behave like $|\sigma^{(k)} - \sigma_h^{(k)}| \le Ch^m$ for some constant $C > 0$ and some positive integer m. We have $\log\left(|\sigma^{(k)} - \sigma_h^{(k)}|\right) \le \log(C) + m\log(h)$ by taking the logarithm on both sides of the inequality. Thus, the convergence rate m can be approximated by the slopes of the best-fit lines through error data, assuming the log-log data behaves linearly. Figure 3.3 exemplifies this linear behaviour. Note that the figure has the relative error on the y-axis, which affects C but not m. The relative error illustrates the convergence more effectively as the lines are easily distinguished in the figure.

Figure 3.3 includes best-fit lines demonstrating the linear behaviour of the error. The slopes of these lines are all approximately $m = 2$, hence the convergence is *quadratic* in the mesh size. Note that m is the worst-case convergence rate. The figures in the following subsections illustrate this as the slopes of the best-fit lines become steeper and less linear as the mesh becomes very fine.

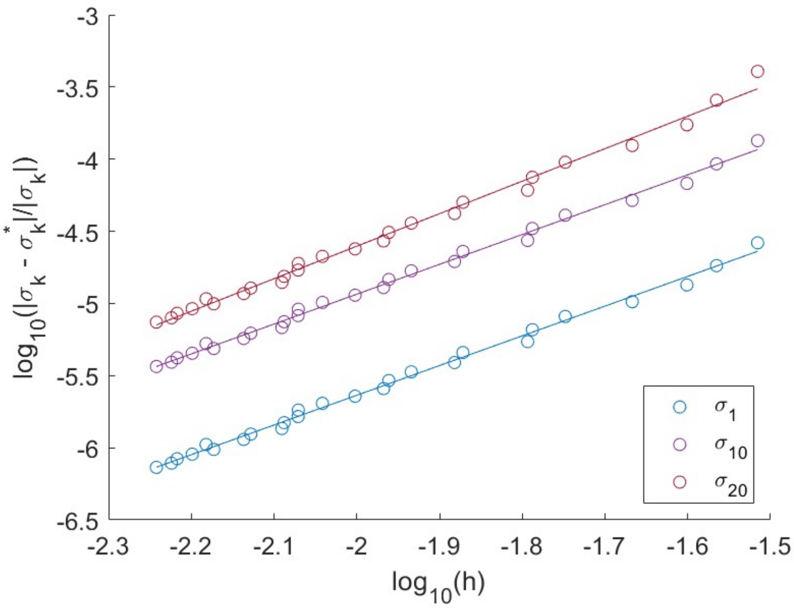


Figure 3.3: Log-log plot of relative error for the Steklov-Laplace problem on the unit disk. The true eigenvalues are denoted by σ_k, the approximations by σ_k^*, and the mesh size by h. The errors are decreasing at least quadratically as the mesh size decreases.

3.3.2 Nonconforming Method

In the nonconforming method, V_h is chosen such that $V_h \not\subseteq X_{\mathrm{T}}(\Omega)$. To implement this in code, we do not include the condition that $\nu \cdot \mathbf{u}_h = 0$ on the boundary of the cube. Recall, the condition that $\mathbf{u}_h$ is tangential is part of the definition of our solution space (2.7) from Section 2.5, $X_{\mathrm{T}}(\Omega) = H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega)$. Since the vectorial Sobolev space (2.6), denoted by $H_0(\mathrm{div}, \Omega)$, includes only tangential vector fields, $X_{\mathrm{T}}(\Omega)$ must too. We claim that the nonconforming method fundamentally fails for the modified Steklov-Maxwell problem, despite involving well-conditioned matrices (for $\alpha \neq 0$) and evidence of convergence.

Ferraresso, Lamberti, & Stratis in [4] derive the exact eigenvalues on the unit ball in $\mathbb{R}^3$ using the theory of spherical harmonics. For $\theta = 1$, the spectrum consists of two sequences of eigenvalues decreasing asymptotically along the line $y = x$. Figure 3.4 suggests that the nonconforming method fails to approximate the eigenvalues, as the computed eigenvalues vary immensely from the exact eigenvalues. In the figure, the approximations (in blue) stay near zero, whereas the true eigenvalues (in green and red) are decaying as the eigenvalue number k increases. That is, our approximations are inaccurate and we must modify our approach. Note that the computed eigenvalues in Figure 3.4 were generated on as fine a mesh as possible (given limited computational power and memory).

We conclude this section with a warning about convergence tests and conditioning. Consider Figure 3.5. This is a convergence plot for the nonconforming method with $\alpha = \theta = 1$, where we compare our best approximation to approximations on coarser meshes. The linear behaviour of the data in this plot illustrates that the approximation error decreases as h tends to zero. Furthermore, the slopes of the best-fit lines are all approximately equal to two, except for the 1st and 5th eigenvalue, so the convergence is quadratic. As previously discussed, Figure 3.4 reveals that the eigenvalues are converging to incorrect numbers despite apparent convergence: while the approximations appear to converge, they are inaccurate.

Lastly, we examine the conditioning of the linear system associated with the nonconforming method. Table 3.1 illustrates that, except for A_2, the condition numbers of the matrices defined in Section 3.1.5 are large (Inf denotes numbers too big to be represented in MATLAB). The last column of the table is relevant since approximating the eigenvalues of the weak modified Steklov-Maxwell problem amounts to solving the generalized eigenvalue problem $Ac = -\lambda Bc$ for $\lambda \in \mathbb{C}$, $c \in \mathbb{C}^n$, where $A = (A_1 - \alpha A_2 + \theta A_3)$. The last column of Table 3.1 demonstrates that A is well-conditioned for $\alpha \neq 0$, as the condition number of A is small. Since A_1 and A_3 have large condition numbers, A_2 makes the system well-conditioned. Hence, a well-conditioned system does not imply accurate results.

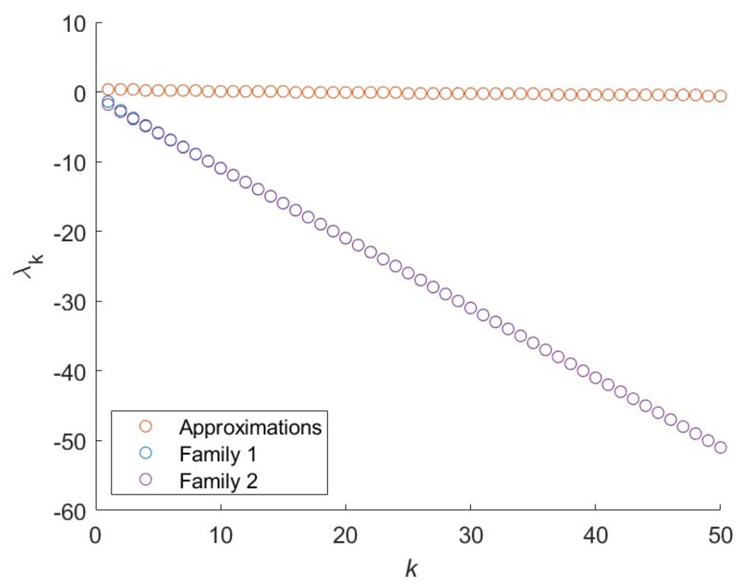


Figure 3.4: The first 50 eigenvalues of the first and second families for the ball compared with the computed eigenvalues for $\alpha = \theta = 1$.

Matrix	A_1	A_2	A_3	B	$A_1 - A_2 + A_3$
Nonconforming Method	1.54e17	50.04	Inf	Inf	6.04e7
Conforming Method	5.00e47	7.80e35	1.20e51	Inf	4.18e33

Table 3.1: Condition numbers of the finite element matrices defined in Section 3.1.5.

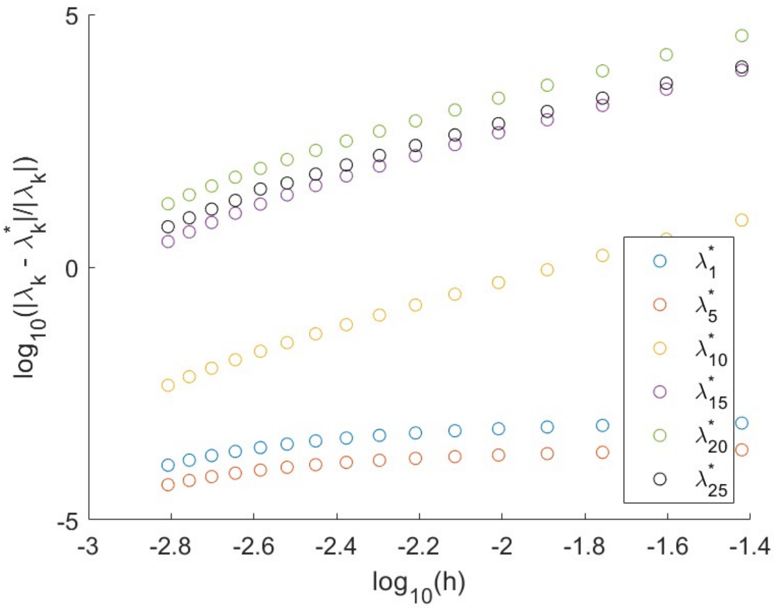


Figure 3.5: Log-log plot of relative error for $\alpha = \theta = 1$ with a nonconforming method. The true k-th eigenvalue is denoted as λ_k, the approximation by λ_k^*, and the mesh size by h.

3.3.3 Conforming Method

In the conforming method, we take V_h to be a proper subspace of $X_{\mathrm{T}}(\Omega)$. Specifically, we search a set smaller than $X_{\mathrm{T}}(\Omega)$ for a solution and increase the size of this set from within $X_{\mathrm{T}}(\Omega)$ as $h \to 0$. To implement this in our code, we manually set $\mathbf{u}_h$ to be tangential: $\mathbf{u}_h \cdot \nu = 0$ on Γ_h.

Given the failure of the nonconforming method, our first question should be whether our eigenvalues are accurate. One way to do this is to compare the asymptotic behaviour of the eigenvalues for different domains [14]. Since we do not have exact solutions to the modified Steklov-Maxwell problem on the unit cube, we compare the asymptotic behaviour of our approximations to the known asymptotics for the unit ball. Recall that the spectrum for the unit ball consists of two families of eigenvalues. If $\theta = 1$, then each family asymptotically decays along the line $y = x$ (see the columns of Table 3.2). Notice that the eigenvalues for the ball and sphere are very similar in magnitude, so we have some confirmation that the conforming method is more accurate than the nonconforming. We have also verified the conforming method's accuracy by numerically validating Theorem 3.10 from [11]. This result gives necessary and sufficient conditions for when problem (2.8) has a zero eigenvalue. With the question of accuracy addressed, we may now determine if the matrices associated with the conforming method are well-conditioned.

Consider Table 3.1. The condition numbers for the finite element matrices are considerably higher than in the nonconforming case. With large condition numbers, we may expect the matrices and their linear combinations to be ill-conditioned. However, our computations demonstrate consistency among eigenvalues of repeated multiplicity. For example, the first eigenvalue for $\alpha = \theta = 1$ is $\lambda_1 = -1.6521$. MATLAB displays this eigenvalue three times, each occurrence agreeing to twelve decimal places (indicating that λ_1 is of multiplicity three). From Section 3.2, we know that iterative methods compute the eigenvalues; each occurrence of λ_1 in the outputted values corresponds to a new iteration. We expect errors to accumulate in each iteration, and large condition numbers signify that the matrices involved are sensitive to these errors. Since the three occurrences of λ_1 agree to twelve decimal places, the matrices may not be as poorly conditioned as suggested. Thus, the conforming method appears to be consistent.

Finally, we must verify that our approximations converge to the true eigenvalues as the mesh size vanishes. Since we do not know the exact eigenvalues on the cube, we first take as small a mesh size as possible, say h_T, and compute the approximate eigenvalues λ_{h_T} for the mesh Ω_{h_T}. These approximations act as our true eigenvalues since we do not know the exact eigenvalues on the cube. Next, we compute the eigenvalues for various $h > h_T$, denoted λ_h, and compare these approximations to the λ_{h_T}. Table 3.3 and Figure 3.6 provide evidence that $|\lambda_h - \lambda_T| \leq Ch^2$ for some $C > 0$: the eigenvalues converge quadratically as the mesh size decreases. We know (from Section 3.3.1) that the slopes of the best-fit lines capture how quickly the eigenvalues converge. Thus, Table 3.3 signals that the convergence is quadratic. Note that the table has some entries that are below two. For instance, λ_{10} with the parameters $\alpha = 1$, $\theta = 0.1$ seems to have a convergence rate of 1.31. However, Figure 3.6(c) shows that the slope steepens for a smaller mesh size h (to the left of $\log_{10}(h) = -1.1$). The slope of this part is approximately two, and so we still have quadratic convergence for λ_{10}. This behaviour is also seen in the other plots included in Figure 3.6.

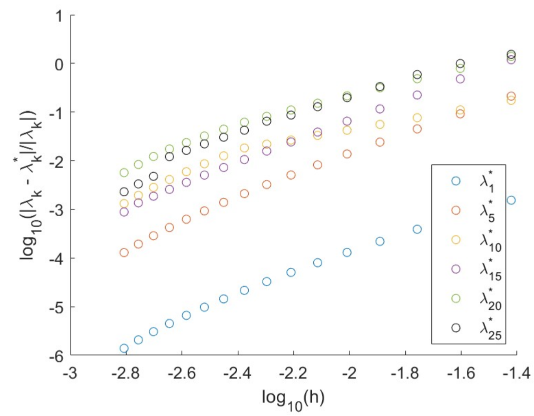
In summary, we have provided evidence of accuracy, consistency, and convergence of the conforming method. Our results indicate that the conforming method is the appropriate approach to computing solutions to the modified Steklov-Maxwell eigenvalue problem using finite elements. More specifically, the failure of the nonconforming method suggests that the tangential boundary condition $\nu \cdot \mathbf{u}_h = 0$ is necessary for our computations to have meaning.

Eigenvalue	λ_1	λ_2	λ_3	λ_4	λ_5
Ball (Family 1)	-1.38	-2.63	-3.73	-4.79	-5.82
Ball (Family 2)	-1.79	-2.85	-3.89	-4.91	-5.92
Cube	-1.65	-2.30	-2.70	-4.32	-4.43

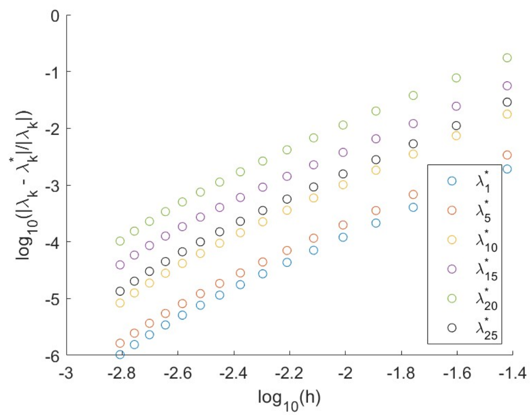
Table 3.2: Eigenvalues for the ball compared with eigenvalues on the cube for $\alpha = \theta = 1$.

Eigenvalue	λ_1	λ_5	λ_{10}	λ_{15}	λ_{20}	λ_{25}
$\alpha = 0.1,\ \theta = 0.1$	2.18	2.29	1.45	2.21	1.66	2.01
$\alpha = 0.1,\ \theta = 1$	2.34	2.37	2.37	2.24	2.31	2.36
$\alpha = 1,\ \theta = 0.1$	2.38	2.32	1.31	2.41	1.65	1.96
$\alpha = 1,\ \theta = 1$	2.34	2.37	2.37	2.25	2.31	2.36
$\alpha = 5,\ \theta = 5$	2.39	2.31	2.04	2.37	2.16	2.29
$\alpha = 10,\ \theta = 10$	2.38	2.31	1.87	2.53	2.03	2.13

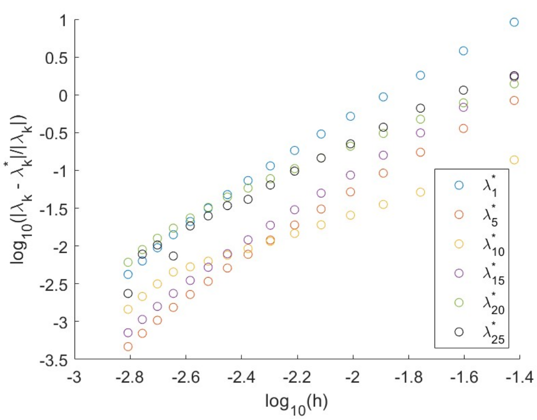
Table 3.3: Slopes of the best-fit lines corresponding to convergence data. The computed slopes demonstrate quadratic convergence in each case. The first four rows correspond to the data displayed in Figure 3.6.



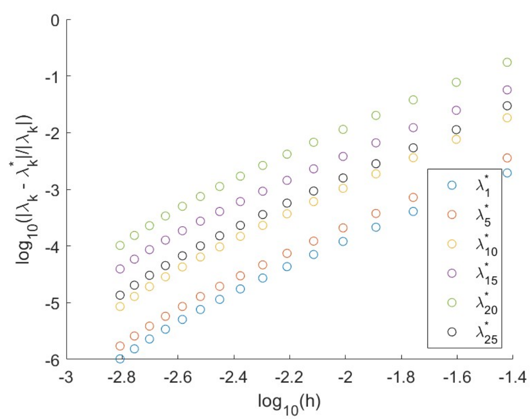
(a) $\alpha = 0.1$, $\theta = 0.1$.



(b) $\alpha = 0.1$, $\theta = 1$.



(c) $\alpha = 1$, $\theta = 0.1$.



(d) $\alpha = 1$, $\theta = 1$.

Figure 3.6: Log-log plots of relative error for different parameters. The true k-th eigenvalue is denoted as λ_k, the approximation by λ_k^*, and the mesh size by h.

Chapter 4

Conclusion

We have established a framework for studying the modified Steklov-Maxwell eigenvalue problem with the finite element method. Our study began with the Steklov eigenvalue problem for the Laplacian. As there is extensive literature on this problem, we wish to understand vectorial analogs of the problem. One such vectorial question involves the curl-curl operator. We have noted the challenges of formulating a Steklov problem for this operator, motivating us to study Lamberti & Stratis's [11] modified Steklov-Maxwell eigenvalue problem. The theoretical framework of the problem and various key results, as described in [4] and [11], were explored in Chapter 2. However, as we are not aware of any numerical studies of the problem, we developed a numerical framework to examine the modified Steklov-Maxwell problem.

We have considered two methods for approximating the eigenvalues of the modified problem using finite elements. The first approach is nonconforming, so the finite element space is not a subspace of $X_{\mathrm{T}}(\Omega)$. The primary concern with this method is that the exact eigenvalues on the unit ball do not agree with our approximations. Our second method is conforming: we look for solutions in a proper subspace of $X_{\mathrm{T}}(\Omega)$. This approach delivers significantly better results. For instance, we have found considerable evidence supporting the convergence of each eigenvalue, regardless of our choice of parameters. Furthermore, the eigenvalue asymptotics on the unit cube seem to agree with the unit ball asymptotics in [4]. The domain invariance of eigenvalue asymptotics is a common characteristic of eigenvalue problems [14], further indicating that the conforming method is the correct approach to numerically studying the modified Steklov-Maxwell problem.

The first inquiry to address in future studies would be to set up a method to approximate the modified Steklov-Maxwell eigenvalues on non-rectangular domains. Since the exact eigenvalues on the unit ball are known, we should first address this case. Our original issue with studying the ball is that there is no direct way to implement the tangential boundary condition in FreeFem++. However, we may be able to force the condition by adding a penalty term. We expect the penalty term to be of the form

$$\beta \int_{\Gamma} (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})\, \mathrm{d}s, \tag{4.1}$$

where $\mathbf{u}, \mathbf{v}$ are in the finite element space, $\mathbf{n}$ is the outward unit normal to Γ, and $\beta > 0$. Suppose we add (4.1) to the left-hand side of the weak formulation (3.4) and take β to be very large. Then we expect $\mathbf{u} \cdot \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{n}$ to become small on Γ so that the equation is balanced. Further experiments are needed to analyze and adjust this method. If this procedure succeeds, we can repeat the convergence tests demonstrated in Section 3.3 for the ball and compare the approximations to the known eigenvalues.

Another question to be further studied is the relation of the modified Steklov-Maxwell problem to the Steklov problem for the curl-curl operator. Given the numerical framework provided in this thesis, we can now study the eigenvalues of (3.4) as α and θ vanish. The problem seemingly approaches the curl-curl problem, but it is not immediately clear that the Steklov-Maxwell eigenvalues tend to the Steklov eigenvalues of the curl-curl operator. Namely, we have provided evidence that if $\lambda(h, \alpha, \theta)$ is an approximation on a mesh of size h to an eigenvalue $\lambda(\alpha, \theta)$ of (3.4), then

$$\lim_{h \to 0} \lambda(h, \alpha, \theta) = \lambda(\alpha, \theta).$$

However, we do not know that

$$\lim_{(\alpha,\theta) \to (0,0)} \lambda(h, \alpha, \theta), \quad \lim_{(\alpha,\theta) \to (0,0)} \lambda(\alpha, \theta)$$

approximate or converge to Steklov eigenvalues of the curl-curl operator. To formulate a well-defined problem for this operator, we must first understand the convergence behaviours as the parameters vanish.

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