MATH-GA 2310 Topology I **Lecture Notes**
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These are lecture notes from a one-semester introductory graduate topology course offered at NYU Courant in the Fall 2024 semester taught by Valentino Tosatti. I may have omitted some proofs done in lecture, but I also occasionally add more detail to some sections, as in the discussion of the Baire category theorem. Some exercises are from the course and others from me, though I became lazy and eventually stopped adding exercises. Perhaps one day I will amend this. There may be typos!

Lecture 1: Introduction, Metric spaces

We begin by studying the topology of $\mathbb{R}^n$, the Euclidean space of n dimensions. Define

$$\mathbb{R}^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}.$$

We know that $\mathbb{R}^n$ is a vector space, $x + y,\ \lambda x \in \mathbb{R}^n$ given $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Here, addition and scalar multiplication works component-wise. We can also equip $\mathbb{R}^n$ with the inner product

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j,$$

so that it is an inner product space (in fact, $\mathbb{R}^n$ is a Hilbert space). The inner product induces a norm,

$$\|x\|^2 := \langle x, x\rangle = \sum_{j=1}^{n} x_j^2.$$

Since $\|\cdot\|$ is a norm, we have that $\|x\| = 0$ if and only if $x = 0$, the origin in $\mathbb{R}^n$. Geometrically, we can interpret the quantity $\|x - y\|$ as the length of the line segment joining x and y - the distance between x and y (though we have not yet shown the line segment minimizes this distance).

One of the most fundamental inequalities in analysis is the Cauchy-Schwarz inequality.

Theorem 1 (Cauchy-Schwarz)**.** For any $x, y \in \mathbb{R}^n$, $|\langle x, y\rangle| \le \|x\|\|y\|$.

Proof. We have

$$0 \le \frac{1}{2}\sum_{i,j=1}^{n}(x_i y_j - x_j y_i)^2 = \sum_{i,j=1}^{n} x_i^2 y_j^2 - \sum_{i,j=1}^{n} x_i y_i x_j y_j = \|x\|^2\|y\|^2 - (\langle x, y\rangle)^2.$$

□

There are various other proofs for Cauchy-Schwarz. One such proof is outlined in the exercises for this lecture.

Corollary (Triangle Inequality)**.** For any $x, y \in \mathbb{R}^n$, we have $\|x + y\| \le \|x\| + \|y\|$.

Proof. This is an immediate consequence of Cauchy-Schwarz by simply expanding out $\langle x + y, x + y\rangle$.

□

As an aside discussion, let us examine the shortest path between two points $x, y \in \mathbb{R}^n$. We define a *curve* in $\mathbb{R}^n$ as $\gamma\colon [a,b] \to \mathbb{R}^n$ where each component γ_j is C^1. The length of γ is defined as

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt.$$

Proposition 1. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a curve and let $x = \gamma(a)$, $y = \gamma(b)$. Then $L(\gamma) \geq \|x - y\|$.

Proof. We may assume without loss that $x \neq y$. Let

$$v = \frac{y - x}{\|y - x\|}.$$

Then $\|v\| = 1$, and

$$\begin{aligned}
\|x - y\| = \langle y - x, v \rangle = \langle \gamma(b) - \gamma(a), v \rangle &= \langle \int_a^b \gamma'(t) \, dt, v \rangle \\
&= \int_a^b \langle \gamma'(t), v \rangle \, dt \\
&\leq \left| \int_a^b \langle \gamma'(t), v \rangle \, dt \right| \\
&\leq \int_a^b \|\gamma'(t)\| \, dt \\
&= L(\gamma),
\end{aligned}$$

using the fundamental theorem of calculus and Cauchy-Schwarz. □

In particular, we are justified in thinking of $\|x - y\|$ as the distance between x and y. If we generalize the notion of distance, we end up with metric spaces.

A *metric space* is a set X and a function $d\colon X \times X \to [0, \infty)$. satisfying

(i) (Symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(ii) (Non-negativity) $d(x,y) \geq 0$ for all $x, y \in X$, with equality holding if and only if $y = x$,

(iii) (Triangle Inequality) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

As previously suggested, $\mathbb{R}^n$ with the distance $d(x,y) := \|x - y\|$ for each $x, y \in \mathbb{R}^n$ is a metric space. We provide some more important examples.

Example (Discrete metric)**.** Let X be any set. Define a metric

$$d(x,y) = \begin{cases} 0 & x = y, \\ 1 & \text{otherwise.} \end{cases}$$

This is the *discrete metric* on a set X. ◇

Example (p-distance in finite dimensions)**.** Given $x, y \in \mathbb{R}^n$, $p \geq 1$, define

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

It is not trivial to show this is a metric. Specifically, the triangle inequality is not trivial, and requires the Minkowski inequality with the counting measure. ◇

Example (sup-metric)**.** For $p = \infty$, we may define the sup-metric (or L^∞-distance) as

$$d_\infty(x,y) = \max_{i=1,\dots,n} |x_i - y_i|$$

for any $x, y \in \mathbb{R}^n$. ◇

Example (General L^p spaces)**.** Let (X, Σ, μ) be a measure space. Define

$$L^p(X) \coloneqq \{f \colon X \to \mathbb{R} : f \text{ is measurable, } \int_X |f|^p < \infty\}/\tilde{}$$

where $f\tilde{g}$ if and only if $f = g$ almost everywhere with respect to μ. Define a metric

$$d_p(f,g) = \left(\int_X |f-g|^p\right)^{\frac{1}{p}}.$$

◇

Example (Induced metric)**.** Let (X, d) be a metric space and $Y \subseteq X$. Then $d|_Y : Y \times Y \to [0, \infty)$ gives the *induced metric* on Y. ◇

Lecture 1 Exercises

1. Prove Cauchy-Schwarz by following the proceeding steps:
   (a) Consider any $\lambda \in \mathbb{R}$ fixed. Use the properties of the inner product to expand $\|x + \lambda y\|^2$.
   (b) Note that this is a quadratic polynomial in λ. Examine it's discriminant to prove Cauchy-Schwarz.

   This method works for more general Hilbert spaces.
2. Prove that the inner product and norm defined in this lecture satisfy the axioms of an inner product and norm.
3. Let X be a set and d_1, d_2 two metrics on X. Determine $d_1 + d_2$ and $\max(d_1, d_2)$ are metrics on X.
4. Let $\gamma \colon [a,b] \to \mathbb{R}^n$ be a smooth curve with $x \coloneqq \gamma(a) \neq \gamma(b) \eqqcolon y$. Show that if its length $L(\gamma)$ is equal to $\|x - y\|$, then γ parametrizes the line segment joining x and y.
5. Let $X \coloneqq (0, \infty)$. Consider $\rho \colon X \times X \to (0\infty)$ defined by

   $$\rho(x,y) \coloneqq \left|\frac{1}{x} - \frac{1}{y}\right|$$

   for all $x, y \in X$. Show that ρ is a metric on X.

Lecture 2: Metric spaces ctd. (continuity, open sets), Topological spaces

On each metric space, we can define the notion of "metric balls". Let (X, d) be a metric space. Given $x \in X$, $r > 0$ the *open ball* of radius r with center x is defined as

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

This definition agrees with our usual notion of disks in $\mathbb{R}^2$. Likewise, a *closed ball* of radius r around x is

$$\overline{B}_r(x) = \{y \in X : d(x, y) \leq r\}.$$

We can use these balls to redefine continuity of a function on a metric space. Let (X, d_X), (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is called *continuous* at $x \in X$ if for every $\epsilon > 0$ there exists $\delta_x > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta_x$. In other words, f is continuous at x if for every $\epsilon > 0$ there is $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq B_\epsilon(f(x))$.

Example. If d_X and d_Y are both the discrete metric, then every f is continuous. ◇

The notion of "open balls" lets us define a more general notion of "openness". We call a set $U \subseteq X$ in the metric space (X, d) *open* if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$. More generally, if $A \subseteq X$ is every set and some open ball around x is contained in A, we call x an *interior point* of A. The set of all interior points of A is denoted by A°. Thus, an open set is any set satisfying $U = U^\circ$. By convention, we take $\varnothing$ to be open.

Example. The open balls $B_r(x)$ in a metric space are open. Indeed, let $y \in B_r(x)$ and take $s = r - d(x, y)$. Then $B_s(y) \subseteq B_r(x)$, as if $z \in B_s(y)$,

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r.$$

◇

Proposition 2 (Properties of Open Sets)**.** Let (X, d) be a metric space. Then

(i) $\varnothing$ and X are open,

(ii) if $U_1, \dots, U_N$ are open, then $\bigcap_{i=1}^N U_i$ is open,

(iii) if $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is any collection of open sets, then $\bigcup_{\alpha \in \mathcal{I}} U_\alpha$ is open.

Proof. (i) is trivial.

For (ii), let $x \in \bigcap_{i=1}^N U_i$. Then there are $r_i > 0$ such that $B_{r_i}(x) \subseteq U_i$ for each $i = 1, \dots, N$. Let $r = \min(r_1, \dots, r_N)$. Then $B_r(x) \subseteq \bigcap_{i=1}^N U_i$.

For (iii), let $x \in \bigcup_{\alpha \in \mathcal{I}} U_\alpha$. Then there exists $\beta \in \mathcal{I}$ such that $x \in U_\beta$. Since U_β is open, $B_r(x) \subseteq U_\beta \subseteq \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ for some $r > 0$. □

We can also rephrase continuity in terms of open sets.

Proposition 3. Let (X, d_X), (Y, d_Y) be metric spaces. Then $f\colon X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every open set $U \subseteq Y$.

Proof. Let f be continuous and $U \subseteq Y$ open. Let $x \in f^{-1}(U)$. Then there exists $r > 0$ such that $B_r(f(x)) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_r(f(x)) \subseteq U$. Thus, $B_\delta(x) \subseteq f^{-1}(U)$.

Conversely, let $\epsilon > 0$ and $x \in X$ be given. We know that $B_\epsilon(f(x)) \subseteq Y$ is open, so $f^{-1}(B_\epsilon(f(x))) \subseteq X$ is open and contains x. Let $\delta > 0$ be such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$. Then $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. □

This equivalent definition of continuity seems quite general, as it does not explicitly involve the metric on X. At some point in the 20th century, people saw this proposition and realized they could abstract it further. This is the start of topology.

A *topological space* is a pair $(X, \mathcal{T})$ consisting of a set X and a *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$, which is a collection of subsets of X satisfying

(i) $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$,

(ii) if $U_1, \ldots, U_N \in \mathcal{T}$, then $\bigcap_{i=1}^N U_i \in \mathcal{T}$,

(iii) for any collection $\{U_\alpha\}_{\alpha \in T} \subseteq \mathcal{T}$, it follows that $\bigcup_{\alpha \in \mathcal{I}} U_\alpha \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. Notice these are exactly the properties we showed in Proposition 2.

Example. If (X, d) is a metric space, then

$$\mathcal{T} = \{U \subseteq X : U \text{ is open with respect to the previous definition of open}\}$$

is the *induced metric topology* on X. ◇

Example. If X is any set, then $\mathcal{T} = \mathcal{P}(X)$ is called the *discrete topology* on X - every set in X is open. ◇

Example. Let X be any set and $\mathcal{T} = \{\varnothing, X\}$. We call $\mathcal{T}$ the *indiscrete (trivial) topology* on X. ◇

Example. Let $X = \mathbb{Q}$ and p be a prime number. We define the p-adic topology. We define $\|\cdot\|_p \colon \mathbb{Q} \to \mathbb{R}$ as follows. Let $\|0\|_p = 0$. If $q \in \mathbb{Q}$, then factor q as $q = p^k \frac{a}{b}$ where $a, b \in \mathbb{Z}$ are not divisible by q and $k \in \mathbb{Z}$. Define $\|q\| = p^{-n}$. The induced metric topology is called the *p-adic topology.* ◇

Lecture 2 Exercises

1. Let X be a set and d the discrete metric on X. Show that the induced metric topology on X is the discrete topology.

2. Let X be a set.

   (a) Let $\mathcal{T}_f$ be all subsets $U \subseteq X$ such that $X \backslash U$ is either finite or all of X. Show that $\mathcal{T}_f$ is a topology on X. *We call this topology the finite complement topology.*

   (b) Is the collection

   $$\mathcal{T}_\infty = \{U \subseteq X : U \backslash X \text{is infinite or empty or all of } X\}$$

   a topology on X?

Lecture 3: Closed sets, Continuity in topological spaces

Last lecture, we defined open sets in a topological space to be the elements of the topology. Another class of important sets in a topological space are the closed sets. We call a set $F \subseteq X$ *closed* with respect to the topology $\mathcal{T}$ on X if $F^c \in \mathcal{T}$. That is, F is closed if and only if F^c is open.

We can characterize closed sets in different ways. Let A be a set. We call a point $p \in A$ a limit point of A if for any open set $U \subseteq X$ containing p there exists a point $q \in A \cap U$ with $q \neq p$. Intuitively, p can be well-approximated by elements of A. We denote the limit points of A by A'.

Example. Given $x \in \mathbb{R}^n$, $r > 0$, any point $y \in \mathbb{R}^d$ satisfying $|x - y| = r$ is a limit point of the open ball $B_r(x)$. To see this, it suffices to show that any open ball around y has non-trivial intersection with $B_r(x)$. Let $\epsilon > 0$ be the supremum of all ϵ^* satisfying $B_{\epsilon^*}(y) \cap B_r(x) = \varnothing$. If $z \in B_\epsilon(y) \cap B_r(x)$, then there exists some $\delta > 0$ such that $B_\delta(z) \subseteq B_\epsilon(y) \cap B_r(x)$. Let L be the line between y and z, parametrized by $sy + (1-s)z$, $s \in [0,1]$. Choose $w \in L$ such that $|w - z| = \delta$. Since y, z, w are co-linear, we have

$$|y - z| = |y - w| + |w - z|,$$

and hence $|y - w| < \epsilon - \delta < \epsilon$, a contradiction. Hence, $B_\epsilon(y) \cap B_r(x)$ is empty. Choose a point z such that z lies on the line segment between x and y and $|y - z| = \epsilon$. Then $|x - z| = r - \epsilon < r$. Hence, we may choose a neighbourhood $B_\delta(z) \subseteq B_r(z)$. Now take $|w - z| = \delta/2$ where w lies on the given line segment. Then $|w - y| = \epsilon - \delta/2 < \epsilon$, a contradiction. Thus, no such ϵ can exist, so the circle bounding each disk consists of limit points of a disk. ◇

Proposition 4. Let X be a topological space and $A \subseteq X$ closed. Then A contains all its limit points.

Proof. Let $x \in A'$. Assume, by way of contradiction, that $x \notin A$. Then A^c is an open set containing x that intersects trivially with A, a contradiction. □

Note that a set can be open and closed at the same time, such as $\varnothing$ and X. Moreover, we can completely characterize a topology by its closed sets.

We now define a more general notion of continuity. Let X and Y be topological spaces. A function $f\colon X \to Y$ is called *continuous* if $f^{-1}(U)$ is open in X for every open set U in Y. If f is bijective and f^{-1} continuous, we call f a *homeomorphism* and call X and Y *homeomorphic*. Essentially, X and Y are topologically the same space. Observe that homeomorphism is an equivalence relation on topological spaces.

Example. Let $\mathbb{B}^n := B_1(0) \subseteq \mathbb{R}^n$ and define a map $f\colon \mathbb{B}^n \to \mathbb{R}^n$ by $f(x) = x/(1 - \|x\|)$. Then f is a well-defined continuous map. Define $g\colon \mathbb{R}^n \to \mathbb{B}^n$ by $g(y) = y/(1 + \|y\|)$. Then g is a well-defined continuous map, and it is easy to check that g is the inverse of f. Hence, $\mathbb{B}^n$ is homeomorphic to $\mathbb{R}^n$. ◇

Example (Stereographic Projection). Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ with the induced Euclidean topology. This is called the *n-sphere*. Let $N = (0, \ldots, 0, 1) \in \mathbb{S}^n$. For every $P \in \mathbb{S}^n \backslash \{N\}$, there is a unique point, which we call $f(P)$, that intersects the hyperplane $x_{n+1} = 0$. This defines a map $f\colon \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$, called *stereographic projection.* This map is continuous and has continuous inverse (see exercises). ◇

Example. Define $f\colon [0,1) \to \mathbb{S}^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Then f is continuous (the components are continuous) and bijective. However, f is not a homeomorphism. For each $x \in \mathbb{S}^1$, we can write $x = (\cos(\theta), \sin(\theta))$ for some unique $\theta \in [0, 2\pi)$. Then $g(x) = \theta/(2\pi)$ is the inverse of f, but g is not continuous. For instance, $g^{-1}([0, 1/2))$ is not open in $\mathbb{S}^1$, since $(1, 0) \in g^{-1}([0, 1/2))$ is not an interior point. $\diamond$

Lecture 3 Exercises

1. Show that $\overline{A} = A \cup A'$ for every set A in a topological space X.
2. Show that $\overline{B}_r(x)$ is closed in a metric space (X, d). This justifies the name "closed ball".
3. Show that Proposition 3 holds when we replace "open" with "closed". In particular, show that $f\colon X \to Y$ is continuous if and only if $f^{-1}(A)$ is closed for every $A \subseteq Y$ closed.
4. Find the stereographic projection map and prove that it is a homeomorphism from $\mathbb{S}^n \backslash \{N\}$ to $\mathbb{R}^n$.

Lecture 4: Basic definitions, Bases

We begin by stating some basic definitions in topological spaces.

Let X be a topological space and $A \subseteq X$ a subset.

- We call A *dense* if $\overline{A} = X$.
- The *closure* of A, denoted $\overline{A}$, is the smallest closed set in X that contains A. That is,
$$\overline{A} \coloneqq \bigcap_{A \subseteq B,\ B \text{ closed}} B.$$
- The *interior* of A, denoted A°, is
$$A^\circ \coloneqq \bigcup_{U \subseteq A,\ U \text{ open}} U.$$
- The *boundary* of A is $\partial A \coloneqq \overline{A} \backslash A^\circ$.

Example. $\mathbb{Q}$ is dense in $\mathbb{R}$. ◇

Example. The closure and interior of $[a, b)$ in the Euclidean topology is $[a, b]$ and (a, b), respectively. The boundary of $[a, b)$ is $\{a, b\}$. ◇

Example. $\partial\mathbb{Q} = \mathbb{R}$ in the Euclidean topology. ◇

Observe that ∂A is always closed, as it is the intersection of two closed sets.

Given a topological space X and two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on X, we say that $\mathcal{T}_1$ is *finer* that $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. That is, every set in open set in $(X, \mathcal{T}_2)$ is open in $(X, \mathcal{T}_1)$ - there are fewer gaps in $(X, \mathcal{T}_1)$ compared to $(X, \mathcal{T}_2)$. Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we say that $\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$.

Example. Every topology is finer than the indiscrete topology and coarser than the discrete topology. ◇

Fix a topological space $(X, \mathcal{T})$ and a subcollection of open sets $\mathcal{B} = \{B_\alpha\}_{\alpha \in \mathcal{I}} \subseteq \mathcal{T}$. We call $\mathcal{B}$ a *basis* for the topology $\mathcal{T}$ if for every $x \in X$ and every open set $U \in \mathcal{T}$ containing x, there exists $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subseteq U$.

Example. $\mathcal{B} = \mathcal{T}$ is a basis, though this is not a particularly enlightening example. ◇

Example. If (X, d) is a metric space, than the set of all open balls is a basis for the induced metric topology, i.e.,
$$\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$$
is a basis. ◇

Example. Let X have the discrete topology. Then $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis. ◇

Example. Let X have the indiscrete topology. Then $\mathcal{B} = \{X\}$ is a basis. ◇

Remark. Let $\mathcal{B}$ be a basis for the topological space $(X, \mathcal{T})$. Then every open set in X can be written as a union of basis elements.

Proof. Let $U \subseteq X$ be open. For each $x \in X$, choose an open set $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq U$. Then $U = \bigcup_{x \in U} U_x$. $\square$

Our definition of continuity can also be framed in terms of bases. If we want to check if a function f is continuous, we need to check the preimage under f of all open sets in Y - this can be a lot of open sets! The next proposition says it suffices to just check that the preimage of basis elements are open.

Proposition 5. Let $f\colon X \to Y$ be a mapping of topological spaces. Fix a basis $\mathcal{B}$ of Y. Then f is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

Proof. The "only if" direction is immediate. Let $U \subseteq Y$ be open. Given $x \in X$, there exists $B_x \in BB$ such that $f(x) \in B_x \subseteq U$. By assumption, $V_x = f^{-1}(B_x)$ is open in X, hence $V = \bigcup_{x \in X} V_x$ is open in X. By construction, $V = f^{-1}(U)$. $\square$

We note that there is an analogous result in measure theory for bases of a σ-algebra.

Given a set X and $\mathcal{A}$ a collection of subsets of X, we can define the *topology generated by* $\mathcal{A}$. Let

$$\mathcal{T}_{\mathcal{A}} := \bigcap_{\mathcal{A} \subseteq \mathcal{T}} \mathcal{T},$$

where $\mathcal{T}$ is a topology. Then $\mathcal{T}_{\mathcal{A}}$ is the smallest topology on X containing $\mathcal{A}$. A basis for this topology is

$$\mathcal{B} := \{U_1 \cap \ldots \cap U_N : U_i \in \mathcal{A},\ N \in \mathbb{N}\}.$$

Lecture 4 Exercises

1. Let X be a topological space and $A \subseteq X$ a subset.
   (a) Show that $\partial\partial A \subseteq \partial A$.
   (b) Find an example where $\partial\partial A \neq \partial A$.
   (c) Show that $\partial\partial\partial A = \partial\partial A$.
2. Let (X, d) be a metric space. Show that the following are equivalent.
   - For every $x \in X$ and $r > 0$, we have $\overline{B}_r(x) = \overline{B_r(x)}$.
   - For every $x \neq y \in X$ and $\epsilon > 0$, there is $z \in X$ such that $d(y, z) < \epsilon$ and $d(x, z) < d(x, y)$.

   Find an example where $\overline{B}_r(x) \neq \overline{B_r(x)}$.
3. Let X be a topological space. We call a collection of closed sets $\mathcal{C}$ a *base for the closed sets* if and only if $\{X \backslash C : C \in \mathcal{C}\}$ is a basis of X. Show that the following are equivalent.
   - $\mathcal{C}$ is a base for the closed sets of X.
   - For every $A \subseteq X$ closed and each point $x \notin A$, there exists $C \in \mathcal{C}$ such that $A \subseteq C$ but $x \notin C$.
   - $\bigcap_{C \in \mathcal{C}} C = \varnothing$ and given $C_1, C_2 \in \mathcal{C}$, we have that $C_1 \cup C_2$ is the intersection of some subfamily of $\mathcal{C}$.

Lecture 5: Operations on topological spaces

In this lecture, we begin our discussion of constructing new topological spaces from old ones. The first item of discussion is the subspace topology. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. The *subspace topology* on A is

$$\mathcal{T}|_A := \{U \cap A : U \in \mathcal{T}\}.$$

In other words, a set $V \subseteq A$ is open in the subspace topology if and only if there exists an open set $U \subseteq X$ such that $V = U \cap A$. We sometimes call V *relatively open* in A. An analogous statement holds for closed sets.

Example. Let $X = \mathbb{R}$ with the Euclidean topology and $A = \bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}$ with the subspace topology. Then the subspace topology on A is the discrete topology, as every singleton is open. However, $A \cup \{0\}$ is no longer discrete with the subspace topology. ◇

Theorem 2 (Universal Property for Subspaces)**.** Let X be a topological space and $A \subseteq X$ a subspace. Given a topological space Y and a map $f: Y \to A$, we have that f is continuous if and only if $\iota \circ f: Y \to X$ is continuous, where $\iota: A \hookrightarrow X$ is the inclusion map.

Proof. Given any $U \subseteq X$ open, we have

$$(\iota \circ f)^{-1}(U) = f^{-1}(\iota^{-1}(U)) = f^{-1}(U \cap A),$$

which is open by continuity of f.

Conversely, take $V \subseteq A$ open and write $V = U \cap A = \iota^{-1}(U)$ for some open set $U \subseteq X$. Then

$$f^{-1}(V) = f^{-1}(\iota^{-1}(U)) = (\iota \circ f)^{-1}(U)$$

is open. □

An immediate consequence is that the inclusion map is always continuous.

The next type of topological space we construct is the *coproduct* of a collection of topological spaces. Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ be any collection of topological spaces. Then the coproduct of the X_α is the disjoint union

$$X := \coprod_{\alpha\in\mathcal{I}} X_\alpha = \{(x, \alpha) : x \in X_\alpha,\ \alpha \in \mathcal{I}\}.$$

For each $\alpha \in \mathcal{I}$, there is a natural inclusion map $\iota_\alpha: X_\alpha \to X$. The *coproduct topology* is defined by declaring that $U \subseteq X$ is open if and only if $\iota_\alpha^{-1}(U \cap \iota_\alpha(X_\alpha))$ is open in X_α for every $\alpha \in \mathcal{I}$. That is, the intersection of U with every X_α is open in X_α.

Theorem 3 (Universal Property for Coproducts)**.** Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ and Y be topological spaces and let X be the coproduct topology on the disjoint union X of the X_α. Then $f: X \to Y$ is continuous if and only if $f|_{X_\alpha}: X_\alpha \to Y$ is continuous for every $\alpha \in \mathcal{I}$.

We can also consider the Cartesian products of topological spaces. First, we consider the product

$$X = X_1 \times \ldots \times X_N = \prod_{i=1}^{N} X_i$$

of finitely many topological spaces X_i. That is,

$$X = \{(x_1, \dots, x_n) : x_i \in X_i,\ i = 1, \dots, N\}.$$

Let

$$\mathcal{B} = \{U_1 \times \dots \times U_N : U_i \subseteq X_i \text{ is open},\ i = 1, \dots, N\}$$

be the collection of "open boxes". The *product topology* on X is the topology generated by $\mathcal{B}$. If $U_1, \dots, U_N$, $V_1, \dots, V_N$ are open sets with $U_j, V_j \subseteq X_j$ for each j, then

$$(U_1 \times \dots \times U_N) \cap (V_1 \times \dots \times V_N) = (U_1 \cap V_1) \times \dots \times (U_N \cap V_N).$$

Thus, $\mathcal{B}$ is closed under taking intersections, and it follows that $\mathcal{B}$ is a basis for the product topology: open sets in X are precisely the union of boxes.

Theorem 4 (Universal Property for Finite Products)**.** Given topological spaces $X_1, \dots, X_N, Y$, a map $f \colon Y \to \prod_{i=1}^N X_i$ is continuous if and only if for each $i = 1, \dots, N$, the map $\pi_i \circ f \colon Y \to X_i$ is continuous, where $\pi_i \colon \prod_{i=1}^N X_i \to X_i$ is the natural projection map.

Proof. Given $i \in \{1, \dots, N\}$ and $U \subseteq X_i$ open, then $X_1 \times \dots \times U \times \dots \times X_N$ is open in the product topology. Hence,

$$f^{-1}(X_1 \times \dots \times U \times \dots \times X_N) = f^{-1}(\pi_i^{-1}(U)) = (\pi \circ f)^{-1}(U)$$

is open.

Conversely, since the boxes are a basis of the product topology, it suffices to show the claim for any box. Let $U_1 \times \dots \times U_N$ be given, where each $U_i \subseteq X_i$ is open. We have that $y \in f^{-1}(U_1 \times \dots \times U_N)$ if and only if $(\pi_i \circ f)(y) \in U_i$ for each $i = 1, \dots, N$. Hence,

$$y \in (\pi_1 \circ f)^{-1}(U_1) \cap \dots \cap (\pi_N \circ f)^{-1}(U_N),$$

which is open in Y. □

This immediately gives that the projection map is continuous.

Lecture 5 Exercises

1. Prove Theorem 3.

2. Let X be a topological space and $A \subseteq X$ a subspace with the subspace topology $\mathcal{T}$. Show that the subspace topology is the coarsest topology such that the inclusion map $\iota \colon A \hookrightarrow X$ is continuous. That is, if $\mathcal{T}'$ is another topology on A such that $\iota \colon A \to X$ is continuous, then $\mathcal{T} \subseteq \mathcal{T}'$.

3. Let X, Y_1, Y_2 be topological spaces and let $f_1 \colon X \to Y_1$, $f_2 \colon X \to Y_2$ be two functions. Suppose $f \colon X \to Y_1 \times Y_2$ is defined by $f(x) = (f_1(x), f_2(x))$. Show that f is continuous if and only if f_1, f_2 are continuous.

Lecture 6: Operations on topological space ctd., Hausdorff spaces

Now that we have introduced finite products, we can deal with infinite products (we omit the proofs). There are actually two topologies we can define on infinite products. Let us deal with the case analogous to the finite product topology. Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of topological spaces and set $X = \prod_{\alpha\in\mathcal{I}} X_\alpha$. The *product topology* on X is generated by the set

$$\mathcal{B} = \{\prod_{\alpha\in\mathcal{I}} U_\alpha : U_\alpha \subseteq X_\alpha \text{ is open for all } \alpha\in\mathcal{I}, \text{ and } \ U_\alpha = X_\alpha \text{ for all } \alpha\in\mathcal{I} \text{ but finitely many } \alpha\}.$$

The analogous universal property holds. Note that the latter condition is vacuous if $\mathcal{I}$ is finite, so this definition agrees with our earlier definition. If we drop this condition, we end up with the *box topology* on X. However, this does not satisfy the universal property, so we will only use the product topology.

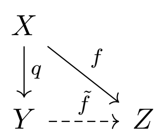
A particularly useful operation on a topological space is taking quotients. Let X, Y be topological spaces and $q\colon X\to Y$ a surjective map. We call q a *quotient map* if $U\subseteq Y$ is open if and only if $q^{-1}(U)\subseteq X$ is open. We immediately see that q is continuous.

Theorem 5 (Universal Property for Quotient Maps)**.** Let $q\colon X\to Y$ be a quotient map. Given a topological space Z and a map $f\colon Y\to Z$, then f is continuous if and only if $f\circ q$ is continuous.

Proof. If $U\subseteq Z$ is open, then $f^{-1}(U)$ is open in Y if and only if $q^{-1}(f^{-1}(U))$ is open in X, by definition of q being a quotient map. □

For a given function $g\colon X\to Y$, the *fibre* of $y\in Y$ under f is the preimage $g^{-1}(\{z\}) = \{x\in X : g(x) = y\}$. In particular, we can think of the fibres of a function as its level sets.

Corollary. Let $q\colon X\to Y$ be a quotient map and $f\colon X\to Z$ a continuous mapping of topological spaces. If f is constant on the fibres of q (i.e. if $f(x) = f(y)$ for all $x, y\in X$ such that $q(x) = q(y)$), then there exists a unique function $\tilde{f}\colon Y\to Z$ such that $f = \tilde{f}\circ q$. That is, $\tilde{f}$ makes the diagram



commute.

Proof. Given $y\in Y$, choose $x\in X$ such that $q(x) = y$. Define $\tilde{f}(y) = f(x)$, so $\tilde{f}$ is well-defined and uniquely given. By the universal property, $\tilde{f}$ is continuous. □

We say that "f passes to the quotient".

We can now look at quotient topologies. Let X be a topological space and $\sim$ an equivalence relation on X. The quotient set $X/\sim$ is the set of equivalence classes of X. Given the natural quotient map $q\colon X\to X/\sim$, where x is mapped to the equivalence class of x, we define the *quotient topology* as $U\subseteq X/\sim$ is open if and only if $q^{-1}(U)$ is open. Then q is a quotient map of topological spaces.

Example. Let $X = [0,1]\subseteq\mathbb{R}$ with the Euclidean topology. Define $x\sim y$ if either $x = y$ or $x = 0$, $y = 1$ or $x = 1$, $y = 0$. That is, we identify 0 and 1 as the same point. We will show later that $X/\sim$ is homeomorphic to $\mathbb{S}^1$.

Note that $\pi\colon X \to X/\sim$ is not an open map, as $\pi([0,1/2))$ is not open in $\mathbb{S}^1$. $\diamond$

Example. Let $X = \overline{B}_1(0) \subseteq \mathbb{R}^2$. Define $\sim$ on X by

$$x \sim y \text{ if and only if } \begin{cases} x_1 = y_1 \\ x_2 = y_2 \end{cases} \quad \text{or} \quad \begin{cases} x_1 = -y_1 \\ x_2 = y_2 \\ (x_1, x_2) \in \partial X \end{cases}.$$

We will show that $X/\sim$ is homeomorphic to $\mathbb{S}^2$. $\diamond$

Example. Let $X = [0,1]^2 \subseteq \mathbb{R}^2$. Define $\sim$ by identifying $(x,0) \sim (x,1)$, $(0,y) \sim (1,y)$ for all $x, y \in [0,1]$. We will show later that $X/\sim$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ (a torus). $\diamond$

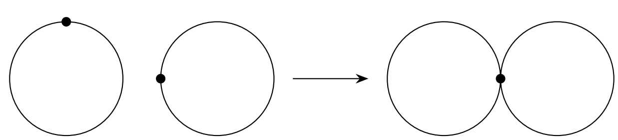
Example (Projective Space)**.** Let $X = \mathbb{R}^{n+1} \backslash \{0\}$ and define $x \sim y$ if $y = \lambda x$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Thus, we can think of $X/\sim$ by the set of lines in $\mathbb{R}^n$ through the origin. We call $\mathbb{RP}^n \coloneqq X/\sim$ the *n-dimensional real projective space*. We can do the same construction to get *n-dimensional complex projective space*, $\mathbb{CP}^n$. A fun fact is that $\mathbb{RP}^1$ and $\mathbb{CP}^1$ are homeomorphic to $\mathbb{S}^1$ and $\mathbb{S}^2$, respectively. $\diamond$

Example (Collapsing of a Subspace)**.** Let X be a topological space and $A \subseteq X$ a subspace. Define $x \sim y$ if $x = y$ or $x, y \in A$. That is, we identify everything in A. Then $X/\sim = (X \backslash A) \coprod \{a\}$ for any $a \in A$. We call this procedure the *collapsing of a subspace*. For instance, if X is a cylinder and A a horizontal cross section, then X/A is the union of two cones connected at a point. $\diamond$

Example (Wedge Sum)**.** Let $X_1, \ldots, X_n$ be topological spaces and $x_j \in X_j$ for each $j = 1, \ldots, n$. Define

$$X \coloneqq \bigvee_{i=1}^{n} X_i = X_1 \vee \ldots \vee X_n = \left(\coprod_{i=1}^{n} X_i \right) /A,$$

where $A = \{(x_1, \ldots, x_n)\}$. We call X the *wedge sum* of the X_i with base points x_i. For instance,



$\diamond$

Example. Let X be a topological space. The *cone over* X is $C(X) \coloneqq (X \times [0,1])/(X \times \{0\})$. That is, we extend X into a "cylinder" then identify all points at the bottom surface. $\diamond$

Overall, quotients give a plethora of ways to construct fascinating and useful topological spaces.

Let us move on from operations on topological spaces to a special class of spaces called Hausdorff spaces. We call a topological space X *Hausdorff* if given $x, y \in X$ there exist open sets $U_x, U_y \subseteq X$ containing x, y, respectively, such that $U_x \cap U_y = \varnothing$. That is, every pair of points in X can be separated by open sets.

Example. X with the indiscrete topology is never Hausdorff. If X instead has the discrete topology, then X is Hausdorff. $\diamond$

Example (Compatibility with Operations).

- If X is Hausdorff and $Y \subseteq X$ is a subspace, then Y is Hausdorff.
- If X, Y are Hausdorff, then $X \coprod Y$ is Hausdorff.
- If X, Y are Hausdorff, then $X \times Y$ is Hausdorff.

$\diamond$

In general, a quotient space is not Hausdorff.

Example (The line with two origins). Consider $\mathbb{R}$ with the Euclidean topology. Then $\mathbb{R} \times \{0,1\} = \mathbb{R} \coprod \mathbb{R}$ is Hausdorff. Define $\sim$ by $(x,0) \sim (x,1)$ for all $x \in \mathbb{R} \backslash \{0\}$. That is, on two copies of $\mathbb{R}$, we identify every point with itself on the other line except for the origin. The space $X = \mathbb{R} \times \{0,1\}/\sim$ is called the *line with two origins*. We claim that X is not Hausdorff. The problem points must be the two origins, $p = (0,0)$ and $q = (0,1)$. Let $\epsilon > 0$ be given and consider the sets

$$A = (-\epsilon, 0) \cup \{(0,0)\} \cup (0, \epsilon), \quad B = (-\epsilon, 0) \cup \{(0,1)\} \cup (0, \epsilon),$$

where $(-\epsilon, 0)$, $(0, \epsilon)$ are intervals. We immediately have that $A \cap B \neq \varnothing$, so we need only show that A and B are open. Let $\pi \colon \mathbb{R} \times \{0,1\} \to X$ be the quotient map. By definition, $\pi^{-1}(A)$ is the product of the open set $(-\epsilon, 0) \cup (0, \epsilon)$ on $\mathbb{R} \times \{1\}$ and the open set $(-\epsilon, \epsilon)$ on $\mathbb{R} \times \{0\}$. Hence, $\pi^{-1}(A)$ is open, and a similar argument gives that $\pi^{-1}(B)$ is open. $\diamond$

We define the *diagonal* of a topological space X in the product space $X \times X$ by

$$\Delta = \{(x,x) : x \in X\} \subseteq X \times X.$$

The diagonal provides a nice characterization of Hausdorff spaces.

Proposition 6. A topological space X is Hausdorff if and only if the diagonal $\Delta \subseteq X \times X$ is closed in the product topology.

We leave the proof as an exercise.

Another interesting example of a space that is not Hausdorff is $\mathbb{R}^n$ with the Zariski topology. The Zariski topology is the natural topology arising in commutative algebra and algebraic geometry when one studies varieties (vanishings) and the prime ideals of a commutative ring. We conclude this lecture by introducing the Zariski topology. The proof of it failing to be Hausdorff is left as an exercise. We stick with $\mathbb{R}^n$, but note that the Zariski topology is much more general. In particular, we can equip the *spectrum* of a ring - the set of prime ideals - with a topology called the Zariski topology. While the construction is similar, familiarity with commutative algebra is more important than in the $\mathbb{R}^n$ case.

First, we note that every polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ can be written as the sum of monomials of the form $x_1^{i_1} \ldots x_n^{i_n}$ where $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$. We define the *zero locus* of $f \in \mathbb{R}[x_1, \ldots, x_n]$ as

$$V(f) \coloneqq \{x \in \mathbb{R}^n : f(x) = 0\}.$$

More generally, if $S \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is a set of polynomials, we define

$$V(S) \coloneqq \{x \in \mathbb{R}^n : f(x) = 0 \text{ for all } f \in S\}.$$

The *Zariski topology* on $\mathbb{R}^n$ is defined by taking the closed sets to be all sets of the form $V(S)$ for some set $S \subseteq \mathbb{R}[x_1, \ldots, x_n]$. We take $V() = \mathbb{R}^n$.

Theorem 6. The Zariski topology satisfies the axioms of a topology.

Proof. We must show that

(i) $\varnothing, \mathbb{R}^n \in \mathcal{T}$;

(ii) if $C_1, \ldots, C_N$ are closed, then $\bigcup_{i=1}^N C_N$ is closed;

(iii) if $\{C_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of closed sets, then $\bigcap_{\alpha \in \mathcal{I}} C_\alpha$ is closed.

(i) We have that $\mathbb{R}^n = V(\varnothing)$ and $\varnothing = V(\mathbb{R}^n)$.

(ii) It suffices to show the claim for just two closed sets C_1, C_2. Write $C_1 = V(S_1),\ C_2 = V(S_2)$ for some $S_1, S_2 \subseteq \mathbb{R}[x_1, \ldots, x_n]$. We claim that $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where

$$S_1 S_2 = \{f_1 f_2 : f_1 \in S_1,\ f_2 \in S_2\}.$$

We immediately have $V(S_1) \cup V(S_2) \subseteq V(S_1 S_2)$. For the converse, take $x \in V(S_1 S_2)$. Choose $g \in S_2$ such that $g(x) \neq 0$ (if no such g exists, the claim is trivial). Then $f(x) = 0$ for all $f \in S_1$, as $(fg)(x) = 0$ for all $f \in S_1$ and $g \neq 0$. Thus, $x \in V(S_1)$.

(iii) We claim that $\bigcap_{\alpha \in \mathcal{I}} V(S_\alpha) = V(\bigcup_{\alpha \in \mathcal{I}} S_\alpha)$, where $C_\alpha = V(S_\alpha)$ for each $\alpha \in \mathcal{I}$. We have $x \in V(\bigcup_{\alpha \in \mathcal{I}} S_\alpha)$ if and only if $f(x) = 0$ for all $f \in S_\alpha$, $\alpha \in \mathcal{I}$. But this is exactly saying that $x \in \bigcap_{\alpha \in \mathcal{I}} V(S_\alpha)$. □

Lecture 6 Exercises

1. Consider $X = \mathbb{R}$ with the Euclidean topology and $A = \mathbb{Q} \subseteq X$ with the subspace topology. Is $Y = X/A$ with the quotient topology Hausdorff?
2. Show that any metric space is Hausdorff.
3. Prove that the product of two Hausdorff spaces is Hausdorff.
4. Prove Proposition 6.
5. Show that $\mathbb{R}^n$ equipped with the Zariski topology is not Hausdorff. *Hint: if* $f \in \mathbb{R}[x_1, \ldots, x_n]$ *satisfies* $f(x) = 0$ *for all* $x \in \mathbb{R}^n$*, then* $f \equiv 0$.

Lecture 7: Countability axioms, Manifolds

In mathematics, it is often convenient to work with countable objects. As we have seen, many important topological spaces are uncountable. However, we can sometimes find a basis for the topology that is countable. This brings us to the idea of countability axioms.

A topological space X is called *second countable* it has a countable basis. That is, there exists a basis of the form $\{U_n\}_{n\in\mathbb{N}}$ where each $U_n \subseteq X$ is open. Every open set $U \subseteq X$ can then be written as $\bigcup_{m\in\mathbb{N}} U_{n_m}$ for some subcollection of $\{U_n\}_{n\in\mathbb{N}}$.

Proposition 7. $\mathbb{R}^n$ is second countable

Proof. We claim that

$$\mathcal{B} = \{B_r(x) : r \in \mathbb{Q}_{>0},\ x \in \mathbb{Q}^n\}$$

is a basis for $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be open. For any $x \in U$, there exists a ball $B_r(x) \subseteq U$ with $r > 0$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we can find $y \in \mathbb{Q}^n$ with $d(x, y) < r/3$. Now pick $s \in \mathbb{Q}$ such that $0 < r/3 < s < r/2$. Let $B = B_s(y) \in \mathcal{B}$. Then $d(x, y) < r/3 < s$, so $x \in B$. If $z \in B_s(y)$, then $d(x, z) \leq d(x, y) + d(z, y) < r$, so $z \in B_r(x)$. Hence, $B_s(y) \subseteq U$. □

We could also take boxes with rational centers and side lengths as a basis of $\mathbb{R}^n$.

There is also a notion of first countability. This is much weaker and more of a "local" countability statement. We say that a topological space X is *first countable* if for any $x \in X$ there exists a countable collection of open sets $\{U_n^x\}_{n\in\mathbb{N}}$, where each U_n^x contains x, such that any open set U containing x also contains some U_n^x. We can think of first countability as giving a local basis, whereas second countability is a global statement (like continuity vs. uniform continuity). Any second countable space is also first countable (see the exercises).

One of the important type of spaces in analysis are those that have a countable dense subset. We call such a space *separable*. More precisely, a topological space X is separable if there exists a countable and dense set $A \subseteq X$. These sets play an important role in functional analysis. For instance, any separable Banach space has a Schauder basis. In any separable Banach space X, we also know that the closed unit ball in X^*, the dual space of X, is sequentially weak-* compact. This is a special case of the Banach-Alaoglu theorem, and is often used in PDE analysis due to its incredibly general statement. At least now, we will not delve too deep into the theory of separable spaces. However, we do have the following nice result.

Theorem 7. Let X be a second countable topological space. Then X is separable.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be the countable basis of X. For each $n \in \mathbb{N}$, choose $x_n \in U_n$ and let $A = \bigcup_{n\in\mathbb{N}} x_n$. Then A is countable. Suppose A is not dense. Then $U = X \backslash \overline{A}$ is a non-empty open set. Then there exists $n \in \mathbb{N}$ such that $x_n \in U_n \subseteq A$, a contradiction. □

Example.

- If $A \subseteq X$ is a subspace of a second countable space X, then A is second countable.
- If $\{X_n\}_{n\in\mathbb{N}}$ is a countable collection of second countable spaces, then $\prod_{n\in\mathbb{N}} X_n$, $\coprod_{n\in\mathbb{N}} X_n$ are second countable.

◇

Example. Suppose X is uncountable and has the discrete topology. Then X is not second countable. Indeed, all singletons $\{x\}$ are open, and any basis of X must contain all the singletons. $\diamond$

We are now ready to introduce manifolds. A set M is a *topological n-manifold* if

(i) M is Hausdorff;

(ii) M is second countable;

(iii) for every $x \in M$, there exists an open set $U \subseteq M$ and a homeomorphism $\phi\colon U \to V$ where $V \subseteq \mathbb{R}^n$ is open in the Euclidean topology.

Manifolds are a special class of topological spaces that are "locally Euclidean".

Example. $\mathbb{R}^n$ is an n-manifold, along with any open set $U \subseteq \mathbb{R}^n$. $\diamond$

Example. $\mathbb{S}^n$ is an n-manifold. To see this, let $x \in \mathbb{S}^n$ and take $\phi\colon \mathbb{S}^n \backslash \{y\} \to \mathbb{R}^n$ a stereographic projection, where $y \neq x$. $\diamond$

Proposition 8. If M is an m-manifold and N an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

The proof is omitted (see Lee's book).

Example. The n-torus $\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}$ is an n-manifold. $\diamond$

Example. Real projective space $\mathbb{RP}^n$ is an n-manifold. We call $[x_0 : x_1 : \ldots : x_n]$, an equivalence classes in $\mathbb{RP}^n$, a *projective coordinate*. Let $x = [x_0 : x_1 : \ldots : x_n]$. Since $x \neq 0$, at least one coordinate x_j of x is non-zero. Thus,

$$x \in U_j \coloneqq \{[x_0 : \ldots : x_n] : x_j \neq 0\}$$

is well-defined. Define a map $\varphi_j\colon U_j \to \mathbb{R}^n$ by

$$\varphi_j(x) \coloneqq \frac{1}{x_j}(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Then φ_j is bijective, with inverse

$$\varphi_j^{-1}(y) \coloneqq [y_1 : \ldots : y_{j-1} : 1 : y_{j+1} : \ldots : y_n].$$

Moreover, φ_j is continuous by the universal property of quotients, since $\hat{\varphi}_j\colon \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^n$ given by

$$\hat{\varphi}_j(x_0, \ldots, x_n) = \frac{1}{x_j}(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

is continuous. A similar construction from $\mathbb{R}^n \to \mathbb{R}^{n+1} \backslash \{0\}$ gives that φ_j^{-1} is continuous. Thus, the third condition is satisfied. We omit checking the other two conditions. $\diamond$

Lecture 7 Exercises

1. (a) Show that second countability implies first countability.

(b) Show that $\mathbb{R}$ with the discrete topology is first countable (we already showed it is not second countable!).

2. Prove that X is a 0-manifold if and only if X is countable and has the discrete topology (recall that $\mathbb{R}^0$ is a singleton).

3. For each $n \in \mathbb{N}$, let $X_n := \mathbb{S}^1$ with a fixed basepoint $x_N \in X_n$, and let

$$X := \bigvee_{n=1}^{\infty} X_n,$$

be the wedge sum of the circles at these basepoints. Prove that X is not second countable. *Hint: show that X is not first countable.*

4. Let X be a second countable space. Show that every collection of disjoint open subsets of X is countable.

5. Show that a metric space is second countable if and only if it is separable.

6. Let M be an n-manifold with boundary. Show that M° is an n-manifold and ∂M an $(n-1)$-manifold (without boundary).

Lecture 8: Connectedness, Path-connectedness,

We have discussed various types of topological spaces like new topological spaces from old ones, Hausdorff spaces, second countable spaces, separable spaces. Another type of topological spaces are connected spaces. We call a topological space X *separated* if there exist non-empty, disjoint open sets $U, V \subseteq X$ such that $X = U \cup V$. That is, X is separated if it is the disjoint union of two open sets. If X is not separated, we call X *connected*. We list some useful properties of connected spaces.

Proposition 9. Let X be a topological space. Then X is connected if and only if $\varnothing$ and X are the only sets that are both open and closed.

Proof. Let A be a set that is both open and closed. Then $X = A \cup (X \backslash A)$. $\square$

The following proposition is particularly useful with $Y = \mathbb{Z}$.

Proposition 10. Let X be connected and Y have the discrete topology. If $f\colon X \to Y$ is continuous, then f is constant.

Proof. Let $x \in X$ be given and set $y = f(x)$. Then $\{y\}$ is open and closed, so that $f^{-1}(\{y\})$ is open, closed, and non-empty. Hence $f^{-1}(\{y\}) = X$. $\square$

We can also prove the intermediate value theorem given a characterization of connected sets in $\mathbb{R}$. First, a more general result.

Proposition 11. Let X, Y be topological spaces and $f\colon X \to Y$ a continuous map. If X is connected, then $f(X) \subseteq Y$ is connected.

Proof. Suppose instead $f(X) = U \cup V$ for non-empty, disjoint open sets $U, V \subseteq Y$. Then $X = f^{-1}(U) \cap f^{-1}(V)$, a contradiction. $\square$

Theorem 8. A subset $A \subseteq \mathbb{R}$ is connected if and only if it is an interval.

Proof. Let A be a connected set in $\mathbb{R}$. We must show that if $a, b \in A$ and $a \leq c \leq b$, then $c \in A$. Suppose instead there are $a, b \in A$ and $a < c < b$ such that $c \notin A$. Then $(-\infty, c) \cap A$ and $[c, \infty) \cap A$ are open in A, disjoint, and their union is equal to A, a contradiction.

Suppose instead A is an interval and separated. Then there are $U, V \subseteq \mathbb{R}$ such that $A \cap U$ and $A \cap V$ separate A. Choose $a \in A \cap U$, $b \in A \cap V$. Without loss of generality, take $a < b$. Then $[a, b] \subseteq A$, as A is an interval. Since U and V are open, there is $\epsilon > 0$ such that $[a, a + \epsilon) \subseteq U \cap A$ and $(b - \epsilon, b] \subseteq V \cap A$. Let $c = \sup(U \cap [a, b])$, so $a + \epsilon \leq c \leq b - \epsilon$, so

$$c \in (a, b) \subseteq A \subseteq U \cup V.$$

If $c \in U$, then $(c - \delta, c + \delta) \subseteq U$ for some $\delta > 0$, which can be chosen such that $\delta < \epsilon$. Then $(c - \delta, c + \delta) \subseteq [a, b]$. Thus, $(c - \delta, c + \delta) \subseteq U \cap [a, b]$. But then $c \geq c + \delta$, which is absurd. Hence, $c \in V$ and $(c - \delta, c + \delta) \subseteq V \cap [a, b]$ for some $\delta < \epsilon$. But then $(c - \delta, c + \delta) \cap (U \cap [a, b]) = \varnothing$, so that $c \leq c - \delta$, a contradiction. $\square$

Corollary (Intermediate Value Theorem)**.** Suppose X is connected and $f\colon X \to \mathbb{R}$ is continuous. Then given $p, q \in X$, f attains every value between $f(p)$ and $f(q)$.

The proof is an immediate consequence of the above two theorems.

Example. $\mathbb{R}$ is not homeomorphic to $[0,\infty)$. Suppose instead it was and let $f\colon [0,\infty) \to \mathbb{R}$ be a homeomorphism. Let $x = f(0)$. Then $f|_{(0,\infty)}\colon (0,\infty) \to \mathbb{R}\backslash\{x\}$ is a homeomorphism, a contradiction. ◇

Some more useful facts:

Proposition 12. Let X be a topological space.

(i) Let $U, V \subseteq X$ open and disjoint, $A \subseteq X$ a connected subspace. If $A \subseteq U \cup V$, then $A \subseteq U$ or $A \subseteq V$.

(ii) If $A \subseteq X$ is a connected subspace, then $\overline{A}$ is connected.

(iii) If $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ is a family of connected subspaces of X with $\bigcap_{\alpha\in\mathcal{I}} A_\alpha \neq \varnothing$, then $\bigcup_{\alpha\in\mathcal{I}} A_\alpha \subseteq X$ is connected.

Proof. (i) Write $A = (A \cap U) \cup (A \cap V)$. Since A is connected, one of these intersections must be empty.

(ii) Suppose $\overline{A} \subseteq U \cup V$, where $U, V \subseteq X$ are open such that $U \cap \overline{A}$, $V \cap \overline{A}$ separate $\overline{A}$. In particular, $U \cap \overline{A}, V \cap \overline{A}$ are closed sets in $\overline{A}$. We have that $A \subseteq \overline{A} \subseteq U \cup V$, so by (i) we may assume $A \subseteq U \cap \overline{A}$. Taking closures, we have $\overline{A} \subseteq U \cap \overline{A}$, since $U \cap \overline{A}$ is closed in the subspace topology on $\overline{A}$. But this is absurd, since $V \cap \overline{A}$ and $U \cap \overline{A}$ are disjoint.

(iii) Suppose $U, V \subseteq X$ are open and separate $A = \bigcup_{\alpha\in\mathcal{I}} A_\alpha$. Then $A \cap U$, $A \cap V$ are non-empty, disjoint open subsets of A and $A \subseteq U \cup V$. Pick $x \in \bigcap_{\alpha\in\mathcal{I}} A_\alpha$. Without loss of generality, we may assume $x \in U$. Then $x \in U \cap A_\alpha$ for each $\alpha \in \mathcal{I}$. By (i), $A_\alpha \subseteq U$ for all $\alpha \in \mathcal{I}$, so that $A \subseteq U$, a contradiction. □

We have introduced a general type of connectedness. However, in spaces like $\mathbb{R}^n$ we often think of another type of connectedness, where we can draw continuous curves between any two points. Such a space is called *path connected*. More precisely, let X be a topological space and $p, q \in X$. A *path* from p to q is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. We call X path connected if there exists a path between any two points of X.

Theorem 9. If X is path connected, then X is connected.

Proof. Fix $p \in X$. Then given any $q \in X$, there exists a path $\gamma_q\colon [0,1] \to X$ from p to q. Let $A_q = \gamma_q([0,1])$. Since $[0,1]$ is connected and γ_q continuous, A_q is connected. Now, $p \in \bigcap_{q\in X} A_q$ and $X = \bigcup_{q\in X} A_q$, so (iii) above gives that X is connected. □

Lecture 8 Exercises

1. Let $X = \mathbb{R}^2\backslash\mathbb{Q}^2$ with the induced Euclidean topology as a subspace of $\mathbb{R}^2$. Show that X is path connected.
2. Let $f\colon \mathbb{S}^1 \to \mathbb{R}$ be continuous. Show that there exists $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.
3. Let X be the line with two origins (Lecture 6), and let $x, y \in X$ be the two origins. Show that X is path connected, but there is no injective path from x to y.

Lecture 9: Path-connectedness ctd., Connected components, Locally connected

We start with some examples of path connected spaces. Note that joining two paths creates a new path.

Example.
- For $n \geq 2$, $\mathbb{R}^n \backslash \{0\}$ is path connected.
- For $n \geq 1$, $\mathbb{S}^n$ is path connected.

$\diamond$

Example (Topologist's sine curve)**.** Let

$$A = \{(x, \sin\left(\frac{1}{x}\right)) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$$

be the *graph* of $\sin(1/x)$ over $(0,1]$. Let $X = \overline{A}$. Then $X = A \cup \{(0,y) : -1 \leq y \leq 1\}$. Note that A is homeomorphic to $(0,1]$ via $\varphi\colon (0,1] \to A$ given by

$$\varphi(x) = (x, \sin(1/x)), \quad \varphi^{-1}(x,y) = x.$$

Since $(0,1]$ is connected, A is connected, and hence $\overline{A}$ is connected. However, $\overline{A}$ is not path connected. Suppose it was. Then there exists a path γ such that $\gamma(0) = (0,0)$ and $\gamma(1) = (1/(2\pi), 0)$. Write $\gamma(t) = (x(t), y(t))$, where x, y are continuous, $x(0) = 0$ and $x(t) > 0$ for all $t > 0$ (without loss of generality), and $y(0) = 0$, $y(t) = \sin(1/x(t))$ for all $t > 0$. Given $n \in \mathbb{N}$, we can find $0 < \tau < x(1/n)$ such that $\sin(1/\tau) = (-1)^n$. By the intermediate value theorem applied to x, there is $0 < t_n < 1/n$ such that $x(t_n) = \tau$. Hence,

$$y(t_n) = \sin(1/x(t_n)) = \sin(1/\tau) = (-1)^n$$

for all $n \in \mathbb{N}$. But y is continuous and $y(0) = 0$, so this is absurd. See Figure 1. $\diamond$

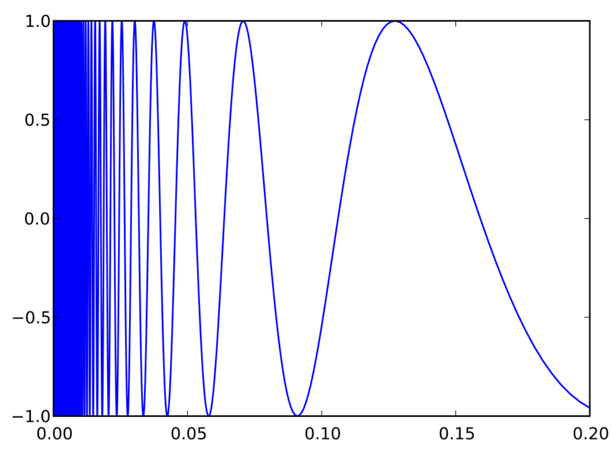


Figure 1: The topologist's sine curve (stolen from Wikipedia).

Now that we have introduced the two major notions of connectedness, let us introduce a convenient interpretation of connectedness. Let X be a topological space and fix $x \in X$. Define

$$\mathcal{C}_x := \{A \subseteq X : x \in A,\ A \text{ is connected}\}.$$

Then $\mathcal{C}_x$ is non-empty, as it contains $\{x\}$. Define

$$C_x := \bigcup_{A \in \mathcal{C}_x} A,$$

the *connected component* of x in X. Note that C_x is actually connected, as each A is connected, and contains x. In fact, it is the largest such set.

Proposition 13. Let X be a topological space and $x, y \in X$. If $C_x \cap C_y \neq \varnothing$, then $C_x = C_y$.

Proof. Let $z \in C_x \cap C_y$. Then $C_x \cup C_y$ is connected, and equal to C_x and C_y by maximality. □

Thus, the connected components partition X.

Remark. Since $\overline{A}$ is connected whenever A is connected, connected components are always closed. However, they may not be open.

Example. Let $X = \mathbb{Q}^2$ with the subspace topology. Then for any $x \in X$, $C_x = \{x\}$. To see this, let $y \in C_x$ and write $x = (a, b)$, $y = (c, d)$ where, without loss of generality, $a < c$. Choose $\lambda \in (a, c) \cup \mathbb{Q}^c$. Let

$$U = \{(\alpha, b) : \alpha < \lambda\},\ V = \{(\alpha, b) : \alpha > \lambda\} \subseteq \mathbb{Q}^2.$$

Then U and V separate C_x, a contradiction. ◇

Similarly, we can define path components P_x, the union of all path-connected subsets of X which contain x. These also partition X, and since path-connectedness implies connectedness, $P_x \subseteq C_x$.

Example. If X is the topologist's sine curve and $x = (0, 0)$, then $C_x = X$ but $P_x = \{(0, y) : -1 \leq y \leq 1\}$. Thus, path components need not be equivalent to connected components. ◇

The notion of connectedness discussed thus far is global. It can sometimes be convenient to have a local description of (path) connectedness. We call a topological space X *locally connected* (resp. *locally path connected*) if for every $x \in X$ and any open set $U_x \subseteq X$ that contains x, there is an open set $V \subseteq U$ such that $x \in V$ and V is connected (resp. path connected). As before, we have that local path-connectedness implied local connectedness.

Example. $\mathbb{R}^n$ with the Euclidean topology is locally path connected, since balls are path connected and form a basis. ◇

Example. The topologist's sine curve is connected but not locally connected, as every ball centered at a point on the y-axis intersected with $\overline{A}$ consists of infinitely many disjoint "segments". ◇

Example. $\mathbb{R} \coprod \mathbb{R}$ is not connected but is locally connected. ◇

Example. Every n-manifold is locally connected. ◇

Proposition 14. If X is locally (path) connected and $x \in X$, then the connected component C_x (P_x) is open.

Proof. If $y \in C_x$, then $y \in U \subseteq X$ for some open and connected set U. By maximality, $C_x \cup U = C_x$. Hence, $U \subseteq C_x$, so y is an interior point of C_x. $\square$

Proposition 15. Let X be locally path connected. Then X is connected if and only if X is path connected.

Proof. We need only show the forward direction. Let $x \in X$. We know that $C_x = X$, and want to show that $P_x = X$ as well. Suppose not and let $V := X \backslash P_x$. Since X is locally path connected, P_x is open, so V is closed. Since X is partitioned by the path components, we can write

$$V = \bigcup_{\alpha \in \mathcal{I}} P_\alpha,$$

where the union does not include P_x. Hence, V is open, a contradiction to X being connected. $\square$

Remark. For a general topological space, C_x is always closed but P_x may not be. For instance, for X the topologist's sine curve and A defined as in the example, any $x \in A$ has $P_x = A$, which is not closed in X.

Lecture 9 Exercises

1. Let X, Y be connected spaces and $A \subsetneq X$, $B \subsetneq Y$ proper subsets. Show that

$$(X \times Y) \backslash (A \times B)$$

is connected.

2. Let X be a topological space and C_x the connected component of $x \in X$. Let C'_x be the intersection of all clopen (closed and open) sets containing x. Show that $C_x \subseteq C'_x$, with equality if X is locally connected.

Lecture 10: Compactness

Our final excursions in point-set topology will look at compact spaces. First, let X be a topological space. A collection of open sets $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ in X is an *open cover* of X if $X = \bigcup_{\alpha\in\mathcal{I}} U_\alpha$. We call X *compact* if any open cover of X admits a finite subcover. More precisely, if $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ is an open cover of X, then there are $U_{\alpha_1}, \dots, U_{\alpha_N} \in \{U_\alpha\}_{\alpha\in\mathcal{I}}$, where $N \in \mathbb{N}$, such that $X = \bigcup_{i=1}^N U_{\alpha_i}$.

Example. Any finite set with any topology is compact. Moreover, any set with the indiscrete topology is compact. $\diamond$

Example. If X has the discrete topology, then X is compact if and only if it is finite. $\diamond$

Proposition 16. Let X, Y be topological spaces, $f\colon X \to Y$ a continuous map. If X is compact, then $f(X) \subseteq Y$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be any open cover of $f(X)$ in Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\mathcal{I}}$ is an open cover of X, hence we can extract a finite subcover $X = \bigcup_{i=1}^N U_{\alpha_i}$. But then $f(X) = \bigcup_{i=1}^N U_{\alpha_i}$. $\square$

Corollary. If X is compact and $q\colon X \to Y$ a quotient map, then $q(Y)$ is compact.

Theorem 10. Given real numbers $a \le b$, the closed interval $[a, b]$ is compact.

Proof. We may assume $a < b$. Let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be any open cover of $[a, b]$. Define

$$X \coloneqq \{x \in (a, b] : [a, x] \text{ is covered by finitely many } U_\alpha\}.$$

We claim that $X \neq \varnothing$. Since the U_α cover $[a, b]$, we can choose $a \in U_1$. Since U_1 is open, $(a, x] \subseteq U_1$ for some $x > a$. Thus, $x \in X$.

Let $c = \sup X \le b$. Then $c \in (a, b]$, and some U_0 in the collection of U_α contains c. Since U_0 is open, there exists $\epsilon > 0$ such that $(c-\epsilon, c] \subseteq U_0$. Now, $c = \sup X$, so there is $x \in (c-\epsilon, c)$, and $[a, x]$ is covered by some $U_1, \dots, U_N$ in $\{U_\alpha\}_{\alpha\in\mathcal{I}}$. Then $U_0, \dots, U_N$ covers $[a, c]$, so that $c \in X$. If $c = b$, then we are done. Assume not. Choose $x \in (c, b)$. Then $[a, x] \subseteq U_0 \cup \ldots \cup U_N$, a contradiction. Thus, $b \in X$. $\square$

We will use this theorem to later prove the Heine-Borel theorem. First, two lemmas. The first of these lemmas is a separation result.

Lemma 1. Suppose X is a Hausdorff space, $A, B \subseteq X$ compact subsets of X with $A \cap B = \varnothing$. Then there are open and disjoint sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$. That is, A and B can be separated by open sets.

Proof. First assume that $B = \{q\}$ is a singleton. Given any $p \in A$, there are disjoint open sets U_p, U_q containing p, q, respectively. Then $\{U_p\}_{p\in A}$ is an open cover of A, hence $A \subseteq \bigcup_{i=1}^N U_{p_i}$ for some $p_1, \dots, p_N \in A$. Then $\bigcup_{i=1}^N U_{p_i}$ and $\bigcap_{i=1}^N V_{p_i}$ are disjoint open sets containing A and B, respectively.

For the general case, the above gives that for each $q \in B$ there are U_q, V_q disjoint open sets in X such that $A \subseteq U_q$ and $q \in V_q$. Choose a finite subcover $V_{q_1}, \dots, V_{q_M}$ of B for some $q_i \in B$. Then $A \subseteq \bigcap_{i=1}^M U_{q_i}$ and $B \subseteq \bigcup_{i=1}^M V_{q_i}$ are disjoint open sets. $\square$

The next lemma provides a useful result for finding neighbourhoods around "lines" in a product.

Lemma 2. Let X be a topological space and $Y \subseteq X$ a compact subspace. Given $x \in X$ and $U \subseteq X \times Y$ open with $\{x\} \times Y \subseteq U$, there exists $x \in V \subseteq X$ open such that $V \times Y \subseteq U$.

Proof. Since open boxes are a basis for $X \times Y$, given $y \in Y$ there are $V_y \subseteq X$, $W_y \subseteq Y$ open such that $(x, y) \in V_y \times W_y \subseteq U$. Now, $\{x\} \times Y$ is homeomorphic to Y, hence compact. Thus, we may choose a finite subcover $\{x\} \times Y \subseteq \bigcup_{i=1}^N V_{y_i} \times W_{y_i}$ for some $y_1, \ldots, y_N \in Y$. Then $V \coloneqq \bigcap_{i=1}^N V_{y_i}$ is open in X, contains x, and

$$V \times Y = V \times \left(\bigcup_{i=1}^N W_{y_i} \right) = \bigcup_{i=1}^N (V \times W_{y_i}) \subseteq \bigcup_{i=1}^N V_{y_i} \times W_{y_i} \subseteq U.$$

□

We can now state some important results for compact spaces.

Theorem 11 (Properties of Compact Spaces/Subsets)**.**

(i) If X is compact, $A \subseteq X$ closed, then A is compact.

(ii) If X is Hausdorff, $A \subseteq X$ compact, then A is closed.

(iii) If (X, d) is a metric space, $A \subseteq X$ compact, then A is bounded.

(iv) If $X_1, \ldots, X_n$ are compact, then $\prod_{i=1}^n X_i$ is compact.

Proof. (i) Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open cover of A. Then $\{U_\alpha\}_{\alpha \in \mathcal{I}} \cup (X \backslash A)$ is an open cover of X. Hence $X = U_1 \cup \ldots \cup U_N \cup (X \backslash A)$ for some U_i in the open cover. Thus, $A \subseteq U_1 \cup \ldots \cup U_N$.

(ii) Given any $x \in X \backslash A$, the first lemma gives that $A \subseteq U$ and $x \in V$ for some disjoint open sets $U, V \subseteq X$. Then $V \cap A = \varnothing$, so A is open.

(iii) Given any $x \in X$, we have $X = \bigcup_{n=1}^\infty B_n(x)$. Extract a finite subcover of A, $A \subseteq \bigcup_{k=1}^m B_{n_k}(x)$ and choose $r = \max(n_1, \ldots, n_m)$. Then $A \subseteq B_r(x)$.

(iv) By induction, it suffices to show the claim for $n = 2$. Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open cover of $X \times Y$. For any $x \in X$, $\{x\} \times Y$ is compact and covered by the U_α, hence covered by some $U_1^x, \ldots, U_{N_x}^x$. By the second lemma, there is some $V_x \subseteq X$ open such that $V_x \times Y \subseteq \bigcup_{i=1}^{N_x} U_i^x$. Now, $X = \bigcup_{x \in X} V_x$, so we extract a finite subcover $V_{x_1}, \ldots, V_{x_m}$ for some $x_i \in X$. Thus,

$$X \times Y = \left(\bigcup_{i=1}^m V_{x_i} \right) \times Y = \bigcup_{i=1}^m V_{x_i} \times Y \subseteq \bigcup_{i=1}^m \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}.$$

□

Lecture 10 Exercises

1. Let X be compact and Y Hausdorff. Show that any injective, continuous function $f \colon X \to Y$ is a homeomorphism between X and its image $f(X)$.

2. Let X be a topological space. A collection of subsets of X, $\mathcal{B} \subseteq \mathcal{P}(X)$, satisfies the *finite intersection property* if for every nonempty finite subcollection $\mathcal{C} \subseteq \mathcal{B}$, we have $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Show that X is compact if and only if every collection of closed subsets of X that satisfies the finite intersection property has non-empty intersection.

Lecture 11: Compactness ctd., Local compactness, Baire Category Theorem

From last lecture, we know that finite products of compact spaces are compact. The same is true for arbitrary products. This is Tychonoff's Theorem.

Theorem 12 (Tychonoff)**.** Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ be compact spaces. Then $\prod_{\alpha\in\mathcal{I}} X_\alpha$ is compact.

The proof is tediuos and is omitted. See Munkres for the proof.

Corollary (Heine-Borel)**.** A subset $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Proof. Since $\mathbb{R}^n$ is Hausdorff, the "'only if" direction follows from the properties of compact subsets. For the converse, since A is bounded we have $A \subseteq [-r, r]^n$ for some $r > 0$. Since $[-r, r]$ is compact in $\mathbb{R}$, the cube $[-r, r]^n$ is compact in $\mathbb{R}^n$. Then A is a closed subset of a compact set, hence compact. □

Corollary. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then f is bounded and attains a global maximum and minimum on X.

Proof. We know that $f(X) \subseteq \mathbb{R}$ is compact, hence closed and bounded. Thus, f is bounded. Moreover, $f(X)$ closed means that $\sup_{x\in X} f(x) \in f(X)$, and similarly $\inf_{x\in X} f(x) \in f(X)$. □

We say that sets $\{A_n\}_{n\in\mathbb{N}}$ are *nested* if $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$.

Proposition 17. Let X be a compact space and $\{F_n\}_{n\in\mathbb{N}}$ a collection of nested, nonempty, closed subsets of X. Then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

Proof. Suppose instead $\bigcap_{n=1}^{\infty} F_n = \varnothing$. Let $U_n = X\backslash F_n$ for each $n \in \mathbb{N}$. Then each U_n is open, $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$, and $X = \bigcup_{n=1}^{\infty} U_n$. Hence, we may extract a finite subcover $X = \bigcup_{j=1}^{N} U_{n_j}$. Then

$$\varnothing = \bigcap_{j=1}^{N} F_{n_j} = F_{n_N} \neq \varnothing,$$

a contradiction. □

Remark. This proposition fails if X is not compact. For instance, take $X = \mathbb{R}$ and $F_n = [n, \infty)$.

Now that we have introduced compactness at the global level, we may study the idea of compactness locally. We call a topological space X *locally compact* if, given $x \in X$ there is a compact set $K_x \subseteq X$ and an open set $U_x \subseteq X$ such that $x \in U_x \subseteq K_x$. That is, every x has an open neighbourhood that is contained in some compact set. This notion is particularly useful when X is Hausdorff. Before stating the lemma, note the following definition. We call a set $A \subseteq X$ *relatively compact* if $\overline{A}$ is compact in X.

Example. $\mathbb{R}^n$ is locally compact. ◇

The main lemma essentially says that we can swap the roles of the compact and open sets in the definition of locally compact spaces, given that the space is Hausdorff. That is, every open neighbourhood of a point in a locally compact Hausdorff space contains a compact neighbourhood of the point. This contrasts with the general definition of locally compact, where every point has some compact neighbourhood that contains an open neighbourhood of the point.

Lemma 3. Let X be a locally compact Hausdorff space. Then, given any $x \in X$ and open set $U \subseteq X$ that contains x, there exists a relatively compact open set $V_x \subseteq X$ such that

$$x \in V_x \subseteq \overline{V_x} \subseteq U.$$

Proof. Since X is locally compact, we can find an open set W and a compact set K in X such that $x \in W \subseteq K$. Since X is Hausdorff, K is closed, so that $K \backslash U$ is also closed. But then $K \backslash U \subseteq K$ is compact. If $K \backslash U = \varnothing$, we have $K \subseteq U$. Hence

$$W \subseteq \overline{W} \subseteq K \subseteq U,$$

as desired. Suppose now that $K \backslash U \neq \varnothing$. Then, since $x \in U$, the compact sets $\{x\}$ and $K \backslash U$ are disjoint. Thus, we may choose disjoint open sets (Lemma 1) $Y, Z \subseteq X$ such that $x \in Y$ and $K \backslash U \subseteq Z$. Define the open set $V_x := Y \cap W$. Then $x \in V_x$, and $V_x \subseteq W$, so that

$$\overline{V_x} \subseteq \overline{W} \subseteq K.$$

In particular, $\overline{V_x}$ is compact. Lastly,

$$V_x \subseteq Y \cap K \subseteq (X \backslash Z) \cap K = K \backslash Z \subseteq U,$$

where the last inclusion is because $K \backslash U \subseteq Z$. Since $K \backslash Z$ is closed, we also have $\overline{V_x} \subseteq U$, as desired. □

We now shift our focus to the Baire Category Theorem (BCT). There are two versions of this theorem, and we will present and prove both versions. The second version also has an equivalent statement, which can be quite useful. The BCT is incredibly useful in functional analysis, providing the foundation for proving the Open Mapping Theorem, Closed Graph Theorem, and Uniform Boundedness Theorem. These incredibly powerful theorems form the basis for much of the subject. We will show that first version of BCT in this lecture, and the second next lecture.

We begin with some basic definitions. Let X be a topological space. A subset $A \subseteq X$ is called a *G_δ set* if it is the countable intersection of open sets. The complement of a G_δ set is called an *F_σ set* - a countable union of closed sets. Before stating the first version of BCT, we have the following remark.

Remark. A set $A \subseteq X$ is dense if and only if for any open set $U \subseteq X$, $A \cap U \neq \varnothing$.

Proof. If $A \cap U = \varnothing$, then $A \subseteq X \backslash U$. But $X \backslash U$ is closed and $\overline{A} = X$, a contradiction. Conversely, if $\overline{A} \neq X$, then $\overline{A} \cap (\overline{A})^c = \varnothing$, □

Theorem 13 (BCT V1)**.** Let X be a locally compact Hausdorff space and $\{U_n\}_{n=1}^{\infty}$ open and dense subsets of X. Then the set $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

We note that the G_δ set $\bigcap_{n=1}^{\infty} U_n$ need not be open.

Proof. Suppose instead $B := \bigcap_{n=1}^{\infty} U_n$ is not dense in X, so that $U := X \backslash \overline{B}$ is non-empty, open, and disjoint from B. Since U_1 is dense, $U \cap U_1 \neq \varnothing$, so by Lemma 3, there is a nonempty, relatively compact open set $W_1 \subseteq \overline{W_1} \subseteq U \cap U_1$. Since U_2 is dense, $W_1 \cap U_2 \neq \varnothing$, so we may choose W_2, a nonempty, relatively compact open set, such that

$$W_2 \subseteq \overline{W_2} \subseteq U_2 \cap W_1 \subseteq \overline{W_1}.$$

Continuing in this way (by induction), we get a collection of nested compact sets $\{\overline{W_n}\}_{n=1}^{\infty}$. By an earlier proposition, $\bigcap_{n=1}^{\infty} \overline{W_n} \neq \varnothing$. However, $\bigcap_{n=1}^{\infty} \overline{W_n} \subseteq \overline{W_1} \subseteq U$ and $\bigcap_{n=1}^{\infty} \overline{W_n} \subseteq \bigcap_{n=1}^{\infty} U_n$, a contradiction to $B \cap U = \varnothing$. $\square$

Lecture 11 Exercises

1. We consider $\mathbb{Q} \subseteq \mathbb{R}$ with the induced Euclidean subspace topology. Use BCT V1 to show that $\mathbb{Q}$ is not locally compact.
2. Show that $\mathbb{R} \backslash \mathbb{Q}$ is not locally compact.

Lecture 12: Baire Category Theorem ctd., Homotopy

The second version of BCT is a statement about metric spaces. We first recall some basic metric space concepts needed to formulate the theorem. Then, we introduce some new topological terminology needed to state an equivalent formulation of BCT V2.

Let (X, d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges* to $x \in X$ if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ whenever $n \geq N$. We call the sequence *Cauchy* if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ whenever $n, m \geq N$.

Remark. Every convergent sequence is Cauchy.

Proof. Let $\epsilon > 0$ be given. Then $d(x_n, x_m) < d(x, x_n) + d(x, x_m) < \epsilon$ for all n, m sufficiently large. □

However, not every Cauchy sequence is convergent.

Example. Let $X = \mathbb{R}\backslash\{0\}$ with the Euclidean metric. Then $\{1/n\}_{n\in\mathbb{N}}$ is Cauchy yet does not converge in X. ◇

A metric space where every Cauchy sequence is also convergent is called *complete*.

Example. $\mathbb{R}^n$ with the Euclidean metric is complete. ◇

Example. Any closed subset of a metric space is complete with the induced subspace metric. ◇

Example. Let $X = C([0, 1])$ with the L^∞ metric (the sup-metric). Then X is complete. ◇

Theorem 14 (BCT V2)**.** Let (X, d) be a nonempty complete metric space and $\{U_n\}_{n=1}^\infty$ be open and dense subsets of X. Then $\bigcap_{n=1}^\infty U_n$ is dense in X.

More generally, a *Baire space* is a space where countable intersections of open and dense sets remain dense. Thus, BCT V1 and V2 say that locally compact Hausdorff spaces and complete metric spaces are two types of topological spaces that are Baire spaces. We will later sketch the proof of BCT V2, as it is, at least initially, very similar to the proof of BCT V1.

The BCT V2 has an elegant equivalent statement, though we need some more theory to state it. Let X be a topological space. We call $A \subseteq X$ *no-where dense* if $(\overline{A})^\circ = \varnothing$ - the closure of A has empty interior. A set of *first category* is any countable union of no-where dense sets. If a set if not of first category, we say it is a set of *second category*. More precisely, if we write $E = \bigcup_{n=1}^\infty E_n$ for some set $E_n \subseteq X$, then E is of second category if some E_n is not no-where dense ($(\overline{E_n})^\circ \neq \varnothing$). With this, we can restate BCT V2 as follows.

Theorem 15 (BCT V2*)**.** Every complete metric space is of the second category.

Proposition 18. The two formulations of the Baire Category Theorem for complete metric spaces are equivalent.

Proof. First suppose BCT V2 holds. Write $X = \bigcup_{n=1}^\infty X_n$ and assume X is of first category. Then each X_n is no-where dense, and, without loss of generality, we may take each X_n to be closed. Then the X_n^c are all open and dense. Hence,

$$\varnothing = X^c = \bigcap_{n=1}^{\infty} X_n^c \neq \varnothing,$$

a contradiction.

Suppose instead BCT V2* holds. Let $U = \bigcap_{n=1}^{\infty} U_n$ and suppose $V \subseteq X$ is any open set. We show $U \cap V \neq \varnothing$, so that U is dense (see the remark in Lecture 11). Consider the complete metric subspace $\overline{V} \subseteq X$. We claim that $U_n \cap V$ is open and dense in the subspace topology on $\overline{V}$. Observe that if $W \subseteq X$ is open, then

$$(W \cap \overline{V}) \cap (U_n \cap V) = (W \cap U_n) \cap V$$

is open in the subspace topology. Each U_n is dense in X, so we also have that $U_n \cap V$ is dense in $\overline{V}$. Now suppose instead

$$U \cap V = \bigcap_{n=1}^{\infty} U_n \cap V = \varnothing.$$

Then in $\overline{V}$, we have

$$\bigcup_{n=1}^{\infty} \overline{V} \backslash (U_n \cap V) = \overline{V}.$$

But each $\overline{V} \backslash (U_n \cap V)$ is no-where dense and closed in $\overline{V}$, so $\overline{V}$ is of first category, a contradictin. $\square$

We now sketch the proof of BCT V2.

Proof of BCT V2 (sketch). Following the same ideas as in the proof of BCT V1, we can find a sequence of points $\{x_n\}_{n \in \mathbb{N}}$ and positive real numbers $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\epsilon_n < 1/n$,

$$\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq \overline{B_{\epsilon_n}(x_n)},$$

and

$$\overline{B_{\epsilon_n}(x_n)} \subseteq U \cap U_1 \cap \ldots \cap U_n,$$

where $U = X \backslash \overline{\cap_{n=1}^{\infty} U_n}$, for all $n \in \mathbb{N}$. We claim that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy. Indeed, for all $m \geq n$, we have $x_m \in \overline{B_{\epsilon_n}(x_n)}$, so $d(x_m, x_n) \leq \epsilon_n < 1/n$. Since X is complete, we can denote the limit of $\{x_n\}$ by $x \in X$. Now,

$$x_m \in \overline{B_{\epsilon_n}(x_n)} \subseteq U \cap U_1 \cap \ldots \cap U_n$$

for all $m \geq n$. Passing to the limit, we find that $x \in U \cap U_1 \cap \ldots \cap U_n$. This holds for arbitrary n, so in fact $x \in U \cap (\bigcap_{n=1}^{\infty} U_n)$, a contradiction. $\square$

Let us examine an examples to see the power of BCT.

Example. Let $f \in \mathbb{R}[x, y]$ be a non-zero real polynomial in x, y. Then $U = \{x \in \mathbb{R}^2 : f(x) \neq 0\}$ is open in the Euclidean topology and dense. If it were not dense, then $\mathbb{R}^2 \backslash \overline{U}$ would be a non-empty open set, so $B_r(x) \subseteq \mathbb{R}^2 \backslash \overline{U}$ for some $x \in \mathbb{R}^2$, $r > 0$. But then $f(z) = 0$ for infinitely many $z \in B_r(x)$, a contradiction.

Now, notice that $\mathbb{Q}[x,y]$ is countable, so we may write $\mathbb{Q}[x,y] = \{f_n\}_{n=1}^{\infty}$. Then

$$U_n = \{x \in \mathbb{R} : f(x) \neq 0\}$$

is open and dense in $\mathbb{R}^2$. By BCT V2, $\bigcap_{n=1}^{\infty} U_n$ is dense in $\mathbb{R}^2$. Thus, we have a dense set of points that do not satisfy any rational polynomial. ◇

This concludes our discussion of BCT. We move from point-set topology to algebraic topology. What is our motivation for introducing algebra into the picture? One basic question is: given two topological spaces X, Y, does there exist $F\colon X \to Y$ a homeomorphism? If there is, it suffices to write down F, if you can. If not, how can we prove it? Algebraic topology says: to any topological space, we associate an algebraic object $\mathscr{A}(X)$ such that

1. $\mathscr{A}$ is, in some sense, computable (i.e. amenable to computations);
2. if X is homeomorphic to Y, then $\mathscr{A}(X) \cong \mathscr{A}(Y)$ (the algebraic structures are isomorphic).

Then it becomes feasible to show that $\mathscr{A}(X) \ncong \mathscr{A}(Y)$, so that X and Y are not homeomorphic. There are various such objects we can study, such as fundamental groups, higher homotopy groups, homology groups, cohomology groups, and much more. In this class, we will only study fundamental groups.

Let us begin by introducing the notion of homotopy. Let X and Y be topological spaces. Fix $I = [0,1]$ We call two continuous maps $f, g\colon X \to Y$ *homotopic* if there is a continuous function $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ for all $x \in X$. We say that F "continuously deforms f to g", and write $f \simeq g$, and call F a *homotopy* between f and g. If g is constant and $f \simeq g$, we will call f *null-homotopic*.

Example. Let $f, g\colon I \to X$ be two paths. Then $f \simeq g$ means that there is $F\colon I \times I \to X$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ for all $x \in [0,1]$. ◇

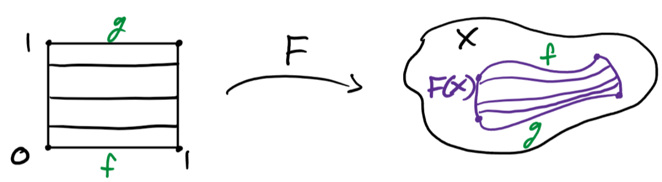


Figure 2: an illustration of homotopy.

If f and p are paths with the same endpoints, i.e., $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, then we call f, g *path homotopic* if there if a continuous map $F\colon I \times I \to X$ such that

$$\begin{cases} F(s,0) = f(s) \\ F(s,1) = g(s) \end{cases} \quad \text{and} \quad \begin{cases} F(0,t) = x_0 \\ F(1,t) = x_1 \end{cases}$$

for all $s, t \in [0,1]$. In this case, for any $t \in I$, $f_t(s) \coloneqq F(s,t)$ is also a path $f_t\colon I \to X$ with initial point at x_0 and final point at x_1. We write $f \simeq_p g$ if f, g are path homotopic.

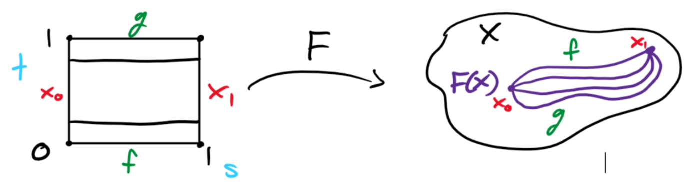


Figure 3: An illustration of path homotopy.

Lecture 12 Exercises

1. The goal of this exercise is to prove the Uniform Boundedness Theorem (UBT) of functional analysis:

   Let $\{T_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of linear operators from a normed linear space X to a complete normed linear space Y (we call Y a *Banach space*).

   The space of all such linear operators is a complete normed linear space, with norm

$$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

   The UBT says that if for any $x \in X$ we have $\sup_{\alpha \in \mathcal{I}} \|T_\alpha x\|_Y = M_x < \infty$, then there is $M > 0$ such that $\sup_{\alpha \in \mathcal{T}} \|T_\alpha\| \leq M$. That is, the pointwise bound over all T_α implies a uniform bound over X. The following steps will guide you in proving this theorem.

   (a) Define
$$S_n := \{x \in X : \|T_\alpha x\| > n \text{ for some } n \in \mathbb{N}\}.$$
   Show that S_n is open. *Hint: Observe that each T_α^{-1} is bounded (hence continuous - you may use this as a fact) and write S_n as the union of open sets.*

   (b) Show that if some S_n is not dense, then the theorem holds.

   (c) Show that if every S_n is dense, then the theorem holds.

2. Use BCT V2 to show that $\mathbb{Q}$ is not a G_δ set.

Lecture 13: Homotopy ctd., Path homotopy

We continue our study of homotopy and path homotopy.

Remark. Suppose X is connected, Y separated, and $f, g\colon X \to Y$ are continuous with images in different components of Y. Then $f \not\simeq g$.

Proof. If instead a homotopy F exists, then $F(X \times I)$ is connected and contains both $f(X)$ and $g(X)$, a contradiction. □

Before continuing, we prove the so-called "Gluing lemma", which will be quite useful in the following results.

Lemma 4 (Gluing Lemma)**.** Let X, Y be topological spaces and $X = \bigcup_{i=1}^{N} X_i$ for closed subspaces X_i. If $f\colon X \to Y$ is a map such that $f|_{X_i}$ is continuous for each $i = 1, \dots, N$, then f is continuous.

Proof. Given $A \subseteq Y$ closed, $f_i^{-1}(A) := f^{-1}(A) \cap X_i$ is closed in X_i by continuity of $f|_{X_i}$. But then $f_i^{-1}(A)$ is closed in X. Then $f^{-1}(A)$ is a finite union of closed sets, hence closed. □

Our first basic result in our study of homotopy is that two maps being homotopic defined an equivalence relation.

Lemma 5. Let X, Y be topological spaces. Then $f \simeq g$ defines an equivalence relation on

$$\{f\colon X \to Y : f \text{ is continuous}\},$$

and $f \simeq_p g$ an equivalence relation on

$$\{f\colon X \to Y : f \text{ is continuous},\ f(0) = x_0,\ f(1) = x_1\}.$$

Proof. We must show reflexivity, symmetry, and transitivity.

(Reflexive) Let f be a given continuous function from X to Y and let $F(x,t) := f(x)$ for all $x \in X$, $t \in I$. Then F is a homotopy (the "constant" homotopy) between f and f. Thus, reflexivity holds.

(Symmetric) Let f, g be given and assume $f \simeq g$. Then there is $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Then defining a homotopy G as $G(x,t) = F(x, 1-t)$ for all $x \in X$, $t \in I$ gives $g \simeq f$.

(Transitive) Let f, g, h be given and assume $f \simeq g$, $g \simeq h$. Pick a homotopy F between f and g and G between g and h. Define

$$H(x,t) := \begin{cases} F(x, 2t) & t \in [0, 0.5], \\ G(x, 2t-1) & t \in [0.5, 1], \end{cases}$$

for all $x \in X$, $t \in I$. Then H is well-defined. Write

$$X \times I = (X \times [0, 0.5]) \cup (X \times [0.5, 1],$$

the union of two closed sets. Then by the Gluing lemma, H is continuous, hence a homotopy between f and h. The path homotopy case is identical.

□

Example. Let $f, g\colon X \to \mathbb{R}^n$ be two continuous maps. Then $f \simeq g$ by the homotopy

$$F(x,t) \coloneqq (1-t)f(x) + tg(x)$$

for all $x \in X, t \in I$. The same F gives that two paths with the same initial and end points are path homotopic.

More generally, if $f, g\colon X \to A$ for some *convex* set $A \subseteq \mathbb{R}^n$, then the same F works. Note that a convex set satisfies $tx + (1-t)y \in A$ for all $x, y \in A$, $t \in [0,1]$. Now, homotopy is an equivalence relation, so in fact all continuous maps into convex sets are null-homotopic. $\diamond$

We note a few definitions before showing more results related to path homotopy. Let X be a topological space. First, for a given path $f\colon I \to X$, the *path class* of f is the equivalence class of f, which we denote by $[f]$. Second, if $f, g\colon I \to X$ are two paths such that $f(0) = x_0$, $f(1) = x_1 = g(0)$, and $g(1) = x_2$, then the *concatenation* of f and g is $h \coloneqq f * g$ defined by

$$h(s) \coloneqq \begin{cases} f(2s) & s \in [0, 0.5], \\ g(2s-1) & s \in [0.5, 1]. \end{cases}$$

Note that h is well-defined and continuous by the gluing lemma. We define $[f * g] \coloneqq [f] * [g]$.

Remark. $[f * g]$ is well-defined.

Proof. Let F, G be homotopies between f, f' and g, g', respectively. Let H be defined as

$$H(s,t) \coloneqq \begin{cases} F(2s, t) & s \in [0, 0.5], \\ G(2s-1, t) & s \in [0.5, 1], \end{cases}.$$

Then H is well-defined, since $F(1,t) = x_1 = G(0,t)$, and continuous by the gluing lemma. Moreover,

$$H(0,t) = F(0,t) = x_0, \quad H(1,t) = G(1,t) = x_2,$$

and

$$H(s,0) = (f * g)(s), \quad H(s,1) = (f' * g')(s),$$

for all $t, s \in I$. Hence, H is a homotopy between $f * g$ and $f' * g'$. $\square$

Thus, concatenation is compatible with the equivalence class structure for path homotopies. With these definitions, we are ready to explore some of the basic properties.

Proposition 19.

(i) If $[f] * ([g] * [h])$ is well-defined, then so is $([f] * [g]) * [h]$. That is, concatenation is associative on path classes.

(ii) Given $x \in X$, let $e_x\colon I \to X$ be the constant map with image x. Let $f\colon I \to X$ be a path with $f(0) = x_0$, $f(1) = x_1$. Then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.

(iii) Given a path $f\colon I \to X$ with $f(0) = x_0$, $f(1) = x_1$. Let $\overline{f}(s) = f(1-s)$ be the *reverse path*. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Proof. (i) For $[f] * ([g] * [h])$ to be defined, we need $g(1) = h(0)$, $f(1) = (g * h)(0) = g(0)$, and for $([f] * [g]) * [h]$ to be defined we need $h(0) = (f * g)(1) = g(1)$ and $f(1) = g(0)$, which are the exact same conditions. To show that they are the same, we must produce a path homotopy between them. Observe that the two paths are reparametrizations of the other. That is, we can find $\varphi\colon I \to I$ such that

$$((f * g) * h)(t) = (f * (g * h)(\varphi(t)),$$

where φ is piecewise linear. Now, we saw that φ is path homotopic to the identity via

$$G(s,t) = (1-t)\varphi(s) + ts.$$

Composing G with $f * (g * h)$ gives the homotopy.

(ii) Observe that, in general, if F is a path homotopy between $f, g\colon I \to X$ and $k\colon X \to Y$ is any continuous map, then $k \circ F$ is a path homotopy between $k \circ f$, $k \circ g$. Also observe that if $f, g\colon I \to X$ are two paths that can be concatenated, i.e. $f(1) = g(0)$, and $k\colon X \to Y$ is a continuous map, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now let $e_0\colon I \to I$ be the constant map with value 0 and $\mathrm{id}\colon I \to I$ the identity. Then $e_0 * \mathrm{id}$ is a path in I from 0 to 1. But I is convex, so there is a path homotopy G between id and $e_0 * \mathrm{id}$. Then $f \circ G$ is a path homotopy in X between $f \circ \mathrm{id} = f$ and

$$f \circ (e_0 * \mathrm{id}) = (f \circ e_0) * (f \circ \mathrm{id}) = e_{x_0} * f.$$

Hence, $[e_{x_0}] * [f] = [f]$. Similarly, let $e_1\colon I \to I$ be the constant path at 1. Then $\mathrm{id} * e_1$ is path homotopic to id, so

$$f \circ (\mathrm{id} * e_1) = (f \circ \mathrm{id}) * (f \circ e_1) = f * e_{x_1}$$

is path homotopic to $f \circ \mathrm{id} = f$. That is, $[f] * [e_{x_1}] = [f]$.

(iii) The reverse of id is $\overline{\mathrm{id}}(s) = 1 - s$, so id can be concatenated with $\overline{\mathrm{id}}$, and $\mathrm{id} * \overline{\mathrm{id}}$, e_0 are two paths beginning and ending at 0. Since I is convex, they are path homotopic. Let H be a path homotopy between e_0 and $\mathrm{id} * \overline{\mathrm{id}}$ in I, so that $f \circ H$ is a path homotopy in X between $f \circ e_0 = e_{x_0}$ and

$$f \circ (\mathrm{id} * \overline{\mathrm{id}}) = f * \overline{f}.$$

Then $[f] * [\overline{f}] = [e_{x_0}]$. Repeating this argument with $\overline{\mathrm{id}} * \mathrm{id}$ gives that $[\overline{f}] * [f] = [e_{x_1}]$. □

Those readers who are familiar with algebra may already have noticed that concatenation seems to be putting some type of algebraic structure on the set of path classes. This observation motivates our study of the so-called "fundamental groups".

Lecture 14: Fundamental groups, Functoriality

Let X be a topological space and fix $x \in X$. A path $f\colon I \to X$ is a *loop* if $f(0) = f(1)$. The *fundamental group* of X based at x is the set

$$\pi_1(X, x) \coloneqq \{f\colon I \to X : f \text{ is a loop with } f(0) = x\}/\simeq_p .$$

With the operation $[f] * [g]$, identity $[e_x]$, and inverse $[\overline{f}$, we see that $\pi_1(X, x)$ is in fact a group. The fundamental group is also called the *first homotopy group.*

Example. Let $X \subseteq \mathbb{R}^n$ be convex and fix $x \in X$. Then $\pi_1(X, x) = \{1\}$, the trivial group. This follows from our earlier observation that all paths with the same initial and end points are path homotopic. ◇

Our definition of the fundamental group requires us to fix a basepoint x. One question then is to examine the algebraic relationship between the fundamental groups at distinct base points. Let $x_0, x_1 \in X$ be given and suppose $\alpha\colon I \to X$ is a path with initial point x_0 and end point x_1. We define a map

$$\begin{aligned} \hat{\alpha}\colon \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_1) \\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha] \end{aligned}$$

(this is the "conjugation by α" map, sending f to its conjugate). Observe that if f is a loop at x_0, then so is $\overline{\alpha} * f * \alpha$.

Proposition 20. The map $\hat{\alpha}$ defined above is an isomorphism of groups.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$ be given. Then

$$\begin{aligned} \hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\overline{\alpha}] * [f] * [\alpha]) * ([\overline{\alpha}] * [g] * [\alpha]) \\ &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= \hat{\alpha}([f] * [g]), \end{aligned}$$

so that $\hat{\alpha}$ is a group homomorphism. Let $\beta = \overline{\alpha}$ and define $\hat{\beta}$ as the conjugation by β map. We leave it as a short exercise to check that $\hat{\alpha}$ and $\hat{\beta}$ are inverses, so that $\hat{\alpha}$ is an isomorphism. ☐

Corollary. If X is path connected, then the fundamental groups at any two points are isomorphic. More generally, for any X and $x_0 \in X$, we have

$$\pi_1(X, x_0) \cong \pi_1(P_{x_0}, x_0) \cong \pi_1(P_{x_0}, x_1)$$

for all $x_1 \in P_{x_0}$, the path component of x_0 in X.

We call a topological space *simply connected* if X is path connected and $\pi_1(X, x_0)$ is trivial for some (hence all) $x_0 \in X$.

Example. $\mathbb{R}^n$, and any convex set $A \subseteq \mathbb{R}^n$ are simply connected. ◇

We can now move on to our discussion of *functoriality.* As the word suggests, the following concepts can be interpreted through the lens of category theory: the fundamental group is a functor from the category of topological spaces along with a base point to the category of groups. However, we will not use this framework, and will stick to a non-category theoretic framework. Given two spaces X, Y with base points $x \in X$, $y \in Y$, we write $h\colon (X,x) \to (Y,y)$ to mean a continuous mapping h such that $h(x) = y$. In this case, if $f\colon I \to X$ is a loop at x in X, then $h \circ f$ is a loop at y in Y. The *homomorphism induced by* h is the map

$$\begin{aligned} h_*\colon \pi_1(X,x) &\longrightarrow \pi_1(Y,y). \\ [f] &\longmapsto [h \circ f] \end{aligned}$$

Proposition 21. The map h_* defined above is a well-defined group homomorphism.

Proof. Suppose $f \simeq_p f'$ via a path homotopy F. Then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$, so $[h \circ f] = [h \circ f']$. Hence, h_* is well-defined. For given loops f, g, we have

$$h_*([f] * [g]) = h_*([f * g]) = [h \circ (f * g)] = [(h \circ f) * (h \circ g)] = h_*([f]) * h_*([g]).$$

□

Example. $\mathrm{id}\colon X \to X$ induces the identity homomorphism $\mathrm{id}_* = \mathrm{id}\colon \pi_1(X,x) \to \pi_1(X,x)$. ⋄

Lemma 6. Given $h\colon (X,x) \to (Y,y)$, $k\colon (Y,y) \to (Z,z)$, we have $(k \circ h)_* = k_* \circ h_*$.

Proof. Given h, k as in the lemma statement,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = k_*(h_*([f])).$$

□

Proposition 22. If $h\colon (X,x) \to (Y,y)$ is a homeomorphism, then $h_*\colon \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism of groups.

Proof. Let $k\colon (Y,y) \to (X,x)$ be the inverse of h. Then by the lemma, $k_* \circ h_*$ and $h_* \circ k_*$ are both the identity homomorphisms. Hence, h_* is an isomorphism. □

This is an incredibly important result, as it tells us that computing the fundamental groups and showing they are not isomorphic is sufficient to conclude the corresponding spaces are not homeomorphic. We can turn a topological question into an algebraic one!

Lecture 15: Computing $\pi_1(X, x_0)$, Retractions, Homotopy equivalence

We showed last time that if we can show two fundamental groups are not isomorphic, then the corresponding map of topological spaces is not a homeomorphism. We still need to actually compute the fundamental groups!

Lemma 7. Let (X, d) be a compact metric space and $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ an open cover of X. Then there exists a *Lebesgue number* $\delta > 0$ such that if $A \subseteq X$ has $\operatorname{diam}(A) < \delta$, then $A \subseteq U_\alpha$ for some $\alpha \in \mathcal{I}$.

Proof. For each $x \in X$, choose α such that $x \in U_\alpha$ and $r_x > 0$ such that $B_{2r_x}(x) \subseteq U_\alpha$. We know $\{B_{r_x}(x)\}_{x \in X}$ is an open cover of X, hence it admits a finite subcover $X = \bigcup_{i=1}^N B_{r_{x_i}}(x_i)$. Let $\delta = \min(r_{x_i})$ and take $A \subseteq X$ with $\operatorname{diam}(A) < \delta$. If $x \in A$, we may take $x \in B_{r_{x_i}}(x_i)$. Now let $y \in A$. Then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + r_{x_i} \leq 2r_{x_i},$$

so $A \subseteq B_{2r_{x_i}}(x_i) \subseteq U_\alpha$. □

We can now find the fundamental group of S^n for $n \geq 2$.

Theorem 16. Let $n \geq 2$. For any $x \in S^n$, the fundamental group $\pi_1(X, x)$ is trivial.

Proof. Let $f: I \to S^n$ be a loop at x. We proceed in two steps.

Step 1: Let $g: I \to S^n$ be another loop at x_0 with g not surjective. We show $f \simeq_p g$. Let $y \in S^n$, $y \neq x$ be given. Let B be an open set containing y but not x (such a B exists, as S^n is Hausdorff). Up to shrinking B, $\overline{B}$ is homeomorphic to $\overline{B}_1(0) \subseteq \mathbb{R}^n$, as S^n is an n-manifold. Let $U = S^n \backslash \{y\}$. Then U is open and $S^n = U \cup B$, so that $I = f^{-1}(B) \cup f^{-1}(U)$ is an open cover of the compact metric space I. By the lemma, there exists a Lebesgue number $\delta > 0$ for this cover. Now let $m \in \mathbb{N}$ be such that $1/m < \delta$ and subdivide I as

$$I = \bigcup_{k=1}^{m} [\frac{k-1}{m}, \frac{k}{m}],$$

so that $\operatorname{diam}([(k-1)/m, k/m]) < \delta$, and $f([(k-1)/m, k/m])$ is a subset of B or U. If for some k we have $f(k/m) = y \notin U$, then $f([(k-1)/m, k/m]), f([k/m, (k+1)/m]) \subseteq B$, so we can just remove k/m. After removing all possible k/m, we have written $I = \bigcup_{i=1}^N I_i$ where the

$$I_i := [a_i, b_i] \subseteq I$$

have disjoint interiors, $f(I_i)$ is a subset of either B or U, and $f(a_i), f(b_i) \neq y$. Now, $\overline{B}\backslash\{y\}$ is homeomorphic to $\overline{B}_1(0)\backslash\{0\}$, which is path connected for $n \geq 2$ (this follows from a similar proof that $\mathbb{R}^n\backslash\{0\}$ is path connected for $n \geq 2$). For each $1 \leq i \leq N$ with $f(I_i) \subseteq B$, there exists a path $g_i: I_i \to B\backslash\{y\}$ with $g_i(a_i) = f(a_i), g_i(a_i) = g_i(b_i)$. Since B is simply connected, $f|_{I_i}$ is path homotopic to g_i inside B. Let $\tilde{f}$ be equal to f on $I\backslash I_i$ and g on I_i. Then $f \simeq_p \tilde{f}$ in S^n. Repeat this for all i and we get a new loop g at x with $g \simeq_p f$ and $y \notin g(I)$.

Step 2: Now, use stereographic projection to conclude that $S^n\backslash\{y\}$ is simply connected, so $f \simeq_p g \simeq_p e_x$. Thus, $\pi_1(X, x)$ is trivial. □

Theorem 17. Let X, Y be topological spaces and $x \in X$, $y \in Y$ basepoints. Then

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y),$$

where the set product on the right is the direct product of groups.

Proof. Given $f\colon I \to X \times Y$, we can write f as $f(t) = (g(t), h(t))$ where $g\colon I \to X$, $h\colon I \to Y$ are maps. Let $p\colon X \times Y \to X$, $q\colon X \times Y \to Y$ be the natural projection maps, so that $p \circ f = g$, $q \circ f = h$. Then f is continuous if and only if g, h are continuous, by the universal property of products. Also, f is a loop at (x, y) is and only if g is a loop at x, h is a loop at y. Thus, a loop in $X \times Y$ is the same as a pair of loops in X and Y. Similarly, if $F\colon I \times I \to X \times Y$ is a path homotopy between two loops f, f' in $X \times Y$ at (x, y), then $G = p \circ F$, $H = q \circ F$ are path homotopies in X and Y between g, g' and h, h', respectively (where g, g', h, h' are defined as above). We can also construct a path homotopy in $F\colon I \times I \to X \times Y$ from path homotopies $G\colon I \to X$, $H\colon I \to Y$. Thus, we get a bijection

$$\begin{aligned} \pi_1(X \times Y, (x, y)) &\longrightarrow \pi_1(X, x) \times \pi_1(Y, y) \\ [f] &\longmapsto [p \circ f, q \circ f]. \end{aligned}$$

Since $p \circ [f * f'] = [p \circ f] * [p \circ f']$, this is a homomorphism. Similarly, the inverse is a homomorphism. Thus, this map is a group isomorphism. □

Example. Let $x \in S^1$ be given. Then the n-torus $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ satisfies

$$\pi_1(\mathbb{T}^n, y) = \underbrace{\pi_1(S^1, x) \times \cdots \times \pi_1(S^1, x)}_{n \text{ times}},$$

where $y = (x, \ldots, x) \in \mathbb{T}^n$. ◇

Let X be a topological space. A subspace $A \subseteq X$ is called a *retract* of X if there is a continuous map $r\colon X \to A$ such that $r \circ \iota = \mathrm{id}_A$ ($r(x) = x$ for all $x \in A$). We call r a *retraction*. Note that r is surjective.

Example. Singletons in a topological space are always retracts. ◇

Example. If $X = Y \times Z$, then $A = Y$ is an adjoint. ◇

Example. $S^n \subseteq \mathbb{R}^{n+1} \backslash \{0\}$ is a retract via $r(x) = x / \|x\|$. ◇

Proposition 23. If $A \subseteq X$ is a retract via $r\colon X \to A$, then for any $x \in A$,

$$\iota_*\colon \pi_1(A, x) \longrightarrow \pi_1(X, x)$$

is injective, and

$$r_*\colon \pi_1(X, x) \longrightarrow \pi_1(A, x)$$

is surjective.

Proof. We have $r \circ \iota = \mathrm{id}_A$, so functoriality gives $r_* \circ \iota_* = \mathrm{id}$. Hence ι_* is injective and r_* is surjective. □

Like topological equivalence via homeomorphism, there is a similar concept for homotopy. We call topological spaces X, Y *homotopy equivalent* if there are continuous maps $f\colon X \to Y$, $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. In this case, we write $X \simeq Y$.

Remark. Being homotopy equivalent is an equivalence relation on topological spaces.

Example. For any $n, m \in \mathbb{N}$, $\mathbb{R}^n \simeq \mathbb{R}^m$. To see this, just take the constant zero maps in both directions and use the structure of $\mathbb{R}^n$ previously discussed. $\diamond$

Lecture 16: Homotopy equivalence ctd.

A topological space X is called *contractible* if it is homotopy equivalent to a point. We call $A \subseteq X$ a *deformation retract* if there is a retraction $r\colon X \to A$ such that $\iota \circ r \simeq \mathrm{id}_X$.

Remark. Since $r \circ \iota_A = \mathrm{id}_A$, $\iota_A \circ r \simeq \mathrm{id}_X$, we see that any deformation retract A of X is homotopy equivalent to X.

Equivalently, there exists a continuous map $H\colon X \times I \to X$ such that $H(x,0) = x$ for all $x \in X$, $H(x,1) \in A$ for all $x \in X$, and $H(a,1) = a$ for all $a \in A$. Given such an H, we take $r(x) \coloneqq H(x,1)$. If instead there exists such an H that also satisfies $H(a,t) = a$ for all $a \in A$ and $t \in I$, we call A a *strong deformation retract* of X. This condition means that points of A do not move along the homotopy.

Example. Let $A = \{a\} \subseteq \mathbb{R}^n$. Then A is a strong deformation retract via $H(x,t) = (1-t)x + ta$. ◇

Example. Let $A = \mathbb{S}^n$ as a subspace of $\mathbb{R}^{n+1}\backslash\{0\}$ or $\overline{B_1}(0)\backslash\{0\}$. Then A is a strong deformation retract via

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}.$$

◇

Now, our goal in discussing homotopy equivalence is to show that any homotopically equivalent spaces have isomorphic fundamental groups. We first need a lemma.

Lemma 8. Let $h, k\colon X \to Y$ be continuous maps which are homotopic via $H\colon X \times I \to Y$. Given $x_0 \in X$, let $y_0 = h(x_0)$, $y_1 = k(x_0)$, and take $\alpha\colon I \to Y$ to be the path $\alpha(t) = H(x_0, t)$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ & \searrow^{k_*} & \downarrow{\hat{\alpha}} \\ & & \pi_1(Y, y_1) \end{array}$$

Proof. Given a loop $f\colon I \to X$ at x_0, we must show that

$$k * [f] = \hat{\alpha}(h * [f]),$$

or equivalently that

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

Define a continuous map $F\colon I \times I \to Y$ by $F(s,t) = H(f(s), t)$. Let

$$\beta_i(s) = (s, i), \quad \gamma_i(t) = (i, t)$$

for $i = 1, 2$. Then $(F \circ \beta_0)(s) = H(f(s), 0) = (h \circ f)(s)$ for all $s \in I$. Similarly,

$$(F \circ \beta_1)(s) = F(s, 1) = H(f(s), 1) = (k \circ f)(s)$$

$$(F \circ \gamma_0)(t) = H(f(0), t) = H(x_0, t) = \alpha(t)$$
$$(F \circ \gamma_1)(t) = H(f(1), t) = H(x_0, t) = \alpha(t)$$

for each $s, t \in I$. Now, $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ with the same initial and end point, so there is a path homotopy $G\colon I \times I \to I \times I$ between them. Then $F \circ G\colon I \times I \to Y$ is a path homotopy between

$$F \circ (\beta_0 * \gamma_1) = (F \circ \beta_0) * (F \circ \gamma_1) = (h \circ f) * \alpha$$

and

$$F \circ (\gamma_0 * \beta_1) = (F \circ \gamma_0) * (F \circ \beta_1) = \alpha * (k \circ f).$$

□

Theorem 18. If $f\colon X \to Y$ is a homotopy equivalence, then for any $x \in X$,

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$$

is a group isomorphism.

Proof. By assumption, there is a continuous map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We have group homomorphisms

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, g(f(x)) \xrightarrow{f_*} \pi_1(Y, f(g(f(x)))).$$

Let $h = g \circ f$, $k = \mathrm{id}_X$, and $x_0 = x$ so that applying the lemma gives that

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{(\mathrm{id}_X)_*} & \pi_1(X, x) \\ & \searrow^{(g \circ f)_*} & \downarrow \hat{\alpha} \\ & & \pi_1(X, g(f(x))) \end{array}$$

is a commutative diagram. By functoriality, $g_* \circ f_* = \hat{\alpha}$ is an isomorphism. Hence, $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is injective, and $g\colon \pi_1(Y, f(x)) \to \pi_1(X, g(f(x)))$ is surjective. Similarly, $f \circ g \simeq \mathrm{id}_Y$, so by the lemma $f_* \circ g_* = \hat{\beta}$, where $\hat{\beta}\colon \pi_1(Y, f(x)) \to \pi_1(Y, f(g(f(x))))$ is an isomorphism. Then g_* is injective, hence an isomorphism. But then $f_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism. □

Lecture 17: Covering spaces

We still do not know that $\pi_1(X, x)$ is not always trivial. To show this, we compute $\pi_1(\mathbb{S}^1, x)$. However, this requires introducing another concept: covering spaces.

Let E, B be two topological spaces and $p\colon E \to B$ a continuous, surjective map. An open set $U \subseteq B$ is called *evenly covered by* p if

$$p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$$

for some collection of open, disjoint sets $V_\alpha \subseteq E$ satisfying $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism for each $\alpha \in \mathcal{A}$. We call $p\colon E \to B$ a *covering space* if p is continuous, surjective, and given $x \in B$, there is an open set $U \subseteq B$ containing x that is evenly covered by p.

Remark. If $p\colon E \to B$ is a covering space, then p is a local homeomorphism.

This remark is an immediate consequence of the definition above, as $p(x)$ is in U for every $x \in E$, and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism where we choose α such that $x \in V_\alpha$ (using that $p^{-1}(U) \subseteq \coprod_\alpha V_\alpha$).

Example. Let B be any topological space and $E = \coprod_{i=1}^N B$ the disjoint union of finitely many copies of B. Then $p\colon E \to B$ with the obvious map is a covering space. $\diamond$

Example. Let $p\colon \mathbb{R} \to \mathbb{S}^1 \subseteq \mathbb{R}^2$ be defined by $p(x) = (cos(2\pi x), \sin(2\pi x))$. We claim that p is a covering space of $\mathbb{S}^1$. Let's first cover $\mathbb{S}^1$ by

$$\begin{aligned} U =& \{(x_1, x_2) \in \mathbb{S}^1 : x_1 > 0\}, \\ & \{(x_1, x_2) \in \mathbb{S}^1 : x_1 < 0\}, \\ & \{(x_1, x_2) \in \mathbb{S}^1 : x_2 > 0\}, \\ & \{(x_1, x_2) \in \mathbb{S}^1 : x_2 < 0\}, \end{aligned}$$

and show each of these is evenly covered by p. We will show this only for U. Observe that

$$p^{-1}(U) = \{x \in \mathbb{R} : \cos(2\pi x) > 0\} = \coprod_{n \in \mathbb{Z}} V_n,$$

where $V_n = (n - 1/4, n + 1/4)$. Now, $p|_{\overline{V_n}}\colon \overline{V_n} \to \overline{U}$ is injective, since $\sin(2\pi x)$ is strictly monotone on $\overline{V_n}$ and surjective since $p^{-1}(x_1, x_2) = n + \arcsin(x_1)/(2\pi)$. Similarly, $p|_{V_n}\colon V_n \to U$ is surjective. Since $\overline{V_n}$ is compact, $p|_{\overline{V_n}}$ is a homeomorphism (see exercises). Hence, $p|_{V_n}$ is also a homeomorphism for each $n \in \mathbb{Z}$. $\diamond$

We now list two nice properties of covering spaces.

Proposition 24. Let $p\colon E \to B$ be a covering space. Then

(i) for any $x \in B$, $p^{-1}(\{x\})$ has the discrete topology,

(ii) p is an open map.

(iii) if $p' \colon E' \to B'$ is another covering space, then $p \times p' \colon E \times E' \to B \times B'$ is a covering space.

Proof. (i) Let $x \in B$ be given and choose an evenly covered open set $U \subseteq B$ that contains x. Write $p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$ for some disjoint open sets $V_\alpha \subseteq E$. Since $p|_{V_\alpha}$ is a homeomorphism, $p^{-1}(\{x\}) \cap V_\alpha$ is a singleton that is open in $p^{-1}(\{x\})$. Hence, $p^{-1}(\{x\})$ has the discrete topology.

(ii) Let $V \subseteq E$ be open and let $x \in p(V)$. Then we can choose an open set $U \subseteq B$ that contains x such that $p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$, as in (i). Pick $y \in V$ such that $p(y) = x$ and $\beta \in \mathcal{A}$ so that $y \in V_\beta$. Then $V_\beta \cap V$ is open in E, hence V_β. But $p|_{V_\beta}$ is a homeomorphism, so $p(V_\beta \cap V)$ is open in U. But then it is also open in B. Hence, p is an open map.

(iii) Given $x \in B$, $x' \in B'$, choose evenly covered $U \subseteq B$, $U' \subseteq B'$ containing x, x', respectively. Write $p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$, $p^{-1}(U') = \coprod_{\beta \in \mathcal{B}} V'_\beta$ for some disjoint open $V_\alpha \subseteq\ B$ and $V_\beta \subseteq\ B'$. Then

$$(p \times p')^{-1}(U \times U') = \coprod_{\alpha,\beta} V_\alpha \times V'_\beta,$$

and $(p \times p')|_{V_\alpha \times V'_\beta} \colon V_\alpha \times V'_\beta \to U \times U'$ is a homeomorphism. □

Example. Consider the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The map $p \colon \mathbb{R}^2 \to \mathbb{T}^2$ is a covering space. This is also true in n-dimensions. ◇

Lemma 9. If $p \colon E \to B$ is a covering space and $B_0 \subseteq B$ a subspace, then $p|_{E_0} \colon E_0 \to B_0$ is a covering space, where $E_0 = p^{-1}(B_0)$.

We leave the proof as an exercise.

Example. Let $p \colon \mathbb{R}^2 \to B$, with $B = \mathbb{S}^1 \times \mathbb{S}^1$, be the usual covering space. Let $x \in S^1$ be fixed and define $B_0 = (\mathbb{S}^1 \times \{x\}) \cup (\{x\} \times \mathbb{S}^1)$. Note that B_0 is homeomorphic to $S_1 \vee S_1$. Now fix $y \in \mathbb{R}$ with $(\cos(2\pi y), \sin(2\pi y)) = x$ (y is unique modulo addition by $\mathbb{Z}$). Then

$$E_0 = p^{-1}(B_0) = \left(\bigcup_{n \in \mathbb{Z}} \mathbb{R} \times \{y+n\} \right) \cup \left(\bigcup_{n \in \mathbb{Z}} \{y+n\} \times \mathbb{R} \right),$$

which is an infinite grid. This follows from the preimage of x under the covering $\mathbb{R} \to \mathbb{S}^1$ being $\bigcup_{n \in \mathbb{Z}} \{y+n\}$. ◇

Lecture 17 Exercises:

1. To be added.
2. Prove Lemma 9.

Lecture 18: Liftings, $\pi_1(\mathbb{S}^1, x)$

Now that we have an understanding of the basic properties of covering spaces, let us discuss the important topic of lifting. Let $p\colon E \to B$ and $f\colon X \to B$ some continuous maps. A *lifting* of f is a continuous map $\tilde{f}\colon X \to E$ such that $f = p \circ \tilde{f}$.

Proposition 25 (Lifting of Paths)**.** Let $p\colon E \to B$ be a covering space and $e_0 \in E$ and take $b_0 \in B$ satisfying $p(e_0) = b_0$. Then given a path $f\colon I \to B$ with $f(0) = b_0$, there is a unique lifting $\tilde{f}\colon I \to E$ such that $\tilde{f}(0) = e_0$.

Proof. Cover B by $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ where each U_α is evenly covered by p. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\mathcal{A}}$ is an open cover of I, so we can find a Lebesgue number $\delta > 0$ for this cover. Choose a subdivision of I, $0 = s_0 < s_1 < \ldots < s_N = 1$ with $s_{i+1} - s + i < \delta$, so that for all i, $f([s_i, s_{i+1}]) \subseteq U_{\alpha(i)}$ for some $\alpha(i) \in \mathcal{A}$. Now we have $f([s_0, s_1]) \subseteq U_{\alpha(0)}$, $p^{-1}(U_{\alpha(0)}) = \coprod_{\beta\in\mathcal{I}} V_\beta$, and $e_0 \in p^{-1}(U_{\alpha(0)})$. Without loss of generality take $e_0 \in V_0$. Then define $\tilde{f}(s \coloneqq (p|_{V_0})^{-1}(f(s))$ for all $s \in [s_0, s_1]$. Then $\tilde{f}$ is continuous from $[s_0, s_1] \to E$ and lifts $f|[s_0, s_1]$, $\tilde{f}(s_0) = e_0$. We can repeat this process finitely many times to obtain a lifting $\tilde{f}\colon I \to E$ with $\tilde{f}(0) = e_0$.

For uniqueness, suppose $\hat{f}$ is another lifting of f with $\hat{f}(0) = e_0$. Then, using the above, we have $p(\hat{f}([s_0, s_1])) = f([s_0, s_1]) \subseteq U_{\alpha(0)}$. Thus, $\hat{f}([s_0, s_1]) \subseteq \coprod_{\beta\in\mathcal{I}} V_\beta$ and $\hat{f}(s_0) = e_0 \in V_0$. But $\hat{f}([s_0, s_1])$ is connected, so $\hat{f}([s_0, s_1]) \subseteq V_0$. Since $p|_{V_0}$ is a homeomorphism, for $s \in [s_0, s_1]$ we have $\hat{f}(s) = ((p|_{V_0})^{-1} \circ f)(s) = \tilde{f}(s)$. Repeating this argument finitely many times gives $\hat{f} = \tilde{f}$. □

Proposition 26 (Lifting of Homotopies)**.** Let $p\colon E \to B$ be a covering space and $p(e_0) = b_0$. Then every continuous map $F\colon I \times I \to B$ with $F(0, 0) = b_0$ has a unique lifting to a continuous map $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0, 0) = e_0$. If F is a path homotopy, then so is $\tilde{F}$.

Proof. By the "lifting of paths" proposition, $F(t, 0)$ has a lifting $\tilde{F}(t, 0)$ and $F(0, s)$ a lifting $\tilde{F}(0, s)$. As before, we subdivide I as

$$0 = s_0 < s_1 < \ldots < s_N = 1,\ 0 = t_0 < t_1 < \ldots < t_M = 1,$$

and let $I_i = [s_{i-1}, s_i]$, $J_j = [t_{j-1}, t_j]$ such that $F(I_i \times J_j)$ is contained in an evenly covered subset of B. We start with $I_1 \times J_1$. We know that F is already continuous on the connected set

$$A \coloneqq (I_1 \times \{0\}) \cup (\{0\} \times J_1).$$

Choose $U \subseteq B$ open and evenly covered such that $F(I_1 \times J_1) \subseteq U$. Then we can take $e_0 = \tilde{F}(0, 0) \in V_0$ and $\tilde{F}(A) \subseteq \coprod_{\beta\in\mathcal{I}} V_\beta$, where $p^{-1}(U) = \coprod_{\beta\in\mathcal{I}} V_\beta$ is the cover. Since $\tilde{F}(A)$ is connected, we have $\tilde{F}(A) \subseteq V_0$. Since $\tilde{F}$ lifts $F|_A$, for any $x \in A$, $p(\tilde{F}(x)) = F(x)$, so that $\tilde{F}(x) = (p|_{V_0})^{-1}(F(x))$, which is continuous by the gluing lemma. We can repeat this for $I_2 \times J_1$, etc. using lexicographic ordering and let A be the union of all previous rectangles and the "bottom" and "left" sides of the current rectangle. Note that A is connected. Repeating this, we get the lift $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0, 0) = e_0$. Uniqueness of $\tilde{F}$ follows from square-by-square uniqueness.

Now if F is a path homotopy, then $F(0, t) = b_0$ for all $t \in I$. Taking the lift $\tilde{F}$, we see that $\tilde{F}(0, t) = p^{-1}(b_0)$, where $p^{-1}(b_0)$ has the discrete topology. Now since I is connected, so is $\tilde{F}(\{0\} \times I)$. Thus, $\tilde{F}(\{0\} \times I)$ is a singleton, which is necessarily b_0. □

The following proposition shows that the lifts between two path homotopic maps are also path homotopic.

Proposition 27. Let $p\colon E \to B$ be a covering space and $p(e_0) = b_0$. Let f, g be two paths from b_0 to b_1 in B and $\tilde{f}, \tilde{g}$ their lifts to paths in E starting at e_0. If $f \simeq_p g$, then $\tilde{f}(1) = \tilde{g}(1)$.

Proof. Let $F\colon I \times I \to B$ be a path homotopy between f, g with $F(0,0) = b_0$ and $\tilde{F}\colon I \times I \to E$ its lift with $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, so for any $t \in I$, $\tilde{F}(0,t) = e_0$ and $\tilde{F}(1,t) = e_1$, for some $e_1 \in E$. Now $\tilde{F}(s,0)$ is a path in E starting at e_0 which lifts $F(s,0) = f(s)$, so uniqueness gives $\tilde{F}(s,0) = \tilde{f}(s)$. Similarly, $\tilde{F}(s,1) = \tilde{g}(s)$ for all $s \in I$, so that

$$\tilde{f}(1) = \tilde{F}(1,0) = e_1 = \tilde{F}(1,1) = \tilde{g}(1).$$

□

We can now link our study of lifting to fundamental groups. Let $p\colon E \to B$ be a covering space with $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lift of f starting at e_0 and define a map $\phi\colon \pi_1(B, b_0) \to p^{-1}(b_0)$ by

$$\phi([f]) = \tilde{f}(1) \in p^{-1}(b_0) \subseteq E.$$

Then ϕ is a well-defined map we call the *lifting correspondence*.

Proposition 28. (i) If E is path connected, then ϕ is surjective.

(ii) If E is simply connected, then ϕ is bijective.

Proof. (i) Since E is path connected, given any $e_1 \in p^{-1}(b_0)$ there is $\tilde{f}\colon I \to E$ with $\tilde{f}(0) = e_0$, $\tilde{f}(1) = e_1$. Then $f = p \circ \tilde{f}$ is a loop in B at b_0 and $\phi([f]) = e_1$.

(ii) Given $[f], [g] \in \pi_1(B, b_0)$ with $\phi([f]) = \phi([g])$, let $\tilde{f}, \tilde{g}$ be their lifts starting at e_0. Then $\tilde{f}(1) = \tilde{g}(1) = \phi([f])$. Since E is simply connected, $\tilde{f} \simeq_p \tilde{g}$, so we can take a path homotopy $\tilde{F}$ between $\tilde{f}$ and $\tilde{g}$. Then $F = p \circ \tilde{F}$ is a path homotopy between f and g. □

Finally, we find a non-trivial fundamental group!

Theorem 19. For any $x \in \mathbb{S}^1$, $\pi_1(\mathbb{S}^1, x) \cong \mathbb{Z}$.

Proof. Take the usual covering space $p\colon \mathbb{R} \to \mathbb{S}^1$ with $e_0 = 0$, $b_0 = p(e_0) = (1,0) \in \mathbb{S}^1$. Then $p^{-1}(b_0) = \mathbb{Z} \subseteq \mathbb{R}$. Now, we know that $\mathbb{R}$ is simply connected, so $\phi\colon \pi_1(\mathbb{S}^1, b_0) \to \mathbb{Z}$ is a bijection. It remains to show ϕ is a group homomorphism. Let $[f], [g] \in \pi_1(\mathbb{S}^1, b_0)$ be given and suppose $\tilde{f}, \tilde{g}$ are their lifts at $0 \in \mathbb{R}$. Let $n = \tilde{f}(1)$, $m = \tilde{g}(1)$ for some $m, n \in \mathbb{Z}$. Then $\phi([f]) = n$, $\phi([g]) = m$. Let $\hat{g}(s) = n + \tilde{g}(s)$ for each $s \in I$. Then $\hat{g}$ is another lift of g, since $p(n + x) = p(x)$ for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$. We have $\hat{g}(0) = n$, so that $\tilde{f} * \tilde{g}$ is a well-defined lift of $f * g$ starting at 0 and ending at $\hat{g}(1) = n + m$. Hence, $\phi([f] * [g]) = \phi([f]) + \phi([g])$. □

Corollary. $\pi_1(\mathbb{T}^n, x) \cong \mathbb{Z}^n$ for each $x \in \mathbb{T}^n$, $n \in \mathbb{N}$.

Lecture 19: Applications and Brouwer Fixed Point Theorem

Knowing the fundamental group of $\mathbb{S}^1$ has numerous consequences.

Theorem 20. There is no retraction of $\overline{B_1(0)} \subseteq \mathbb{R}^2$ onto $\mathbb{S}^1$.

Proof. If a retraction exists, then

$$\iota_*\colon \pi_1(\mathbb{S}^1, x) \hookrightarrow \pi_1(\overline{B_1(0)})$$

is injective, a contradiction. $\square$

Theorem 21 (Brouwer Fixed Point)**.** Let $\overline{B}^2 = \overline{B_1(0)}$. Then any continuous map $f\colon \overline{B}^2 \to \overline{B}^2$ has a fixed point.

Proof. Suppose instead $f(x) \neq x$ for all $x \in \overline{B}^2$. We define a retraction $r\colon \overline{B}^2 \to \mathbb{S}^1$ as follows. Let $\gamma(t) = (1-t)f(x) + tx$ be a line through x and $f(x)$, where $t \in \mathbb{R}$. Then γ intersects $\mathbb{S}^1$ in two places, which we determine explicitly. Solving the quadratic equation

$$1 = \|\gamma(t)\|^2 = (\gamma(t), \gamma(t)) = (1-t)^2\|f(x)\|^2 + t^2\|x\|^2 + 2t(1-t)(f(x), x),$$

which simplifies to

$$t^2(\|x\|^2 + \|f(x)\|^2 - 2(f(x), x)) + 2t((f(x), x) - \|f(x)\|^2) + (\|f(x)\|^2 - 1) = 0.$$

Choose the positive root t_0 and define $r(x) \coloneqq \gamma(t_0)$ for every $x \in \overline{B}^2$. Then r is continuous, since γ is continuous, and t_0 is a continuous function of x (because $f(x) \neq x$ for all x). Moreover, if $x \in \mathbb{S}^1$, then $t_0 = 1$, so $r(x) = \gamma(1) = x$. Thus, r is a retraction, a contradiction. $\square$

Example. Let $x \in \mathbb{R}^n \backslash \{0\}$ for $n \geq 1$. We find $\pi_1(\mathbb{R}^n \backslash \{0\}, x)$. Define

$$f\colon \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{S}^{n-1} \times \mathbb{R}$$

by

$$f(x) = \left(\frac{x}{\|x\|},\ \log(\|x\|)\right).$$

Then f is continuous, with continuous inverse $g(y, t) \coloneqq ye^t$, so f is a homeomorphism. But then

$$\pi_1(\mathbb{R}^n \backslash \{0\}, x) \cong \pi_1(\mathbb{S}^{n-1}, p) \times \pi_1(\mathbb{R}, 0) \cong \pi_1(\mathbb{S}^{n-1}, p) \cong \begin{cases} \mathbb{Z} & \text{n=2,} \\ 0 & \text{otherwise.} \end{cases}$$

$\diamond$

Corollary. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$.

Corollary. Let $A \in \mathbb{R}^{3\times 3}$ be a matrix. If all entries of A are positive, then A has a positive eigenvector.

Proof. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by A and let

$$B = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x^2 + y^2 + z^2 = 1\}$$

be the graph of $\sqrt{1-x^2-y^2}$ over

$$B^1 \coloneqq \{(x,y) \in \mathbb{R}^2 : x, y \geq 0, x^2+y^2 \leq 1\}.$$

Then B is homeomorphic to B^1, and B_1 is homeomorphic to $\overline{B}_1(0) \subseteq \mathbb{R}^2$. Let $\Phi\colon B \to \overline{B}_1(0)$ be a homeomorphism. For given $x \in B$, $T(x)$ has all non-negative components, with one positive. Thus, $\|T(x)\| > 0$ and $f\colon B \to B$ defined by $f(x) = T(x)/\|T(x)\|$ is well-defined and continuous. Hence,

$$\Phi \circ f \circ \Phi^{-1}\colon \overline{B}_1(0) \to \overline{B}_1(0)$$

is continuous. By the Brouwer fixed point theorem, there is $x \in \overline{B}_1(0)$ so that $\Phi(f(\Phi^{-1}(x))) = x$. Define $v = \Phi^{-1}(x) \in B$, so that $f(v) = v$, and $T(v) = \|T(v)\|v$. □

Proposition 29. Let X be a topological space and $f\colon \mathbb{S}^1 \to X$ a continuous map. Then the following are equivalent:

(i) f is null-homotopic,

(ii) f extends to a continuous map $\tilde{f}\colon \overline{B}^2 \to X$,

(iii) $f_*\colon \pi_1(\mathbb{S}^1, x) \to \pi_1(X, f(x))$ is the trivial homomorphism, where we take without loss $x = (1,0)$.

Proof. "(a) $\Longrightarrow$ (b)" Choose $F\colon \mathbb{S}^1 \times I \to X$ continuous such that $F(x,0) = f(x)$ and $F(x,1) = p$ for some $p \in X$ and all $x \in I$. An easy fact is that

$$\begin{aligned} \pi\colon \mathbb{S}^1 \times I &\longrightarrow \overline{B}^2 \\ (y,t) &\longmapsto (1-t)y \end{aligned}$$

is a quotient map. Note that all fibres $\pi^{-1}(x)$, $x \neq 0$, are singletons, and $\pi^{-1}(0) = \mathbb{S}^1 \times \{1\}$. Since F is constant on $\mathbb{S}^1 \times \{1\}$, there is (by a previous lemma) $\tilde{F}\colon \overline{B}^2 \to X$ continuous such that $F = \tilde{F} \circ \pi$. Now, for any $x \in \mathbb{S}^1$, $x = \pi(x, 0)$, so

$$\tilde{F}(x) = \tilde{F}(\pi(x,0)) = F(x,0) = f(x).$$

Thus, $\tilde{F}$ is the desired extension.

"(b) $\Longrightarrow$ (c)" Let $\iota\colon \mathbb{S}^1 \to \overline{B}^2$ be the inclusion map, so $f = \tilde{f} \circ \iota$. Hence $f_* = \tilde{f}_* \circ \iota_*$. But

$$\iota_*\colon \pi_1(S^1, x) \to \pi_1(\overline{B}^2, x) = 0$$

is the trivial homeomorphism, so so is f_*.

"(c) $\Longrightarrow$ (a)" Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the usual covering map and define $p_0 \coloneqq p|_I$, a loop in $\mathbb{S}^1$ at x. We know that $[p_0]$ is a generator of $\pi_1(\mathbb{S}^1, x)$. Since $f * [p_0] = 0$ in $\pi_1(X, f(x))$, the loop $g = f \circ p_0$ in X at $f(x)$ is path homotopic to $e_{f(x)}$ via some $G\colon I \times I \to X$. We have the quotient map

$$p_0 \times \mathrm{id}\colon I \times I \to \mathbb{S}^1 \times I$$

which identifies $\{0\} \times I$ and $\{1\} \times I$ with $\{x\} \times I \subseteq \mathbb{S}^1 \times I$. Since

$$G(0,t) = G(1,t) = f(x)$$

for all $t \in I$, G descends to a continuous map $H\colon \mathbb{S}^1 \times I \to X$, and for all $z \in \mathbb{S}^1$,

$$H(z,0) = H(p_0(y),0) = G(y,0) = g(y) = f(p_0(y)) = f(z),$$

and

$$H(z,1) = H(p_0(y),1) = G(y,1) = f(x),$$

which is constant. Thus, f is null-homotopic.

□

Lecture 19 Exercises

1. Check that the map $p_0 \times \mathrm{id}$ defined in Proposition 29 is indeed a quotient map.

Lecture 20: More tools for computing $\pi_1(X, x)$

We begin with a few more properties of covering spaces and liftings.

Proposition 30. Let $p\colon E \to B$ be a covering space and $p(e_0) = b_0$.

(i) $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

(ii) Let $H \coloneqq p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$. Then the lifting correspondence ϕ induces an injective map $\Phi\colon \pi_1(B, b_0)/H \to p^{-1}(b_0)$. If E is path-connected, then Φ is a bijection.

(iii) Let $f\colon I \to B$ be a loop at b_0 and $\tilde{f}\colon I \to E$ its lift to a path in E beginning at e_0. Then $\tilde{f}$ is a loop if and only if $[f] \in H$.

Remark. We note that $H \subseteq \pi_1(B, b_0)$ need not be a normal subgroup of $\pi_1(B, b_0)$. By $\pi_1(B, b_0)/H$, we just mean the set of left H cosets.

Proof. (i) Let $h\colon I \to E$ be a loop at e_0 with $p_*([h]) = e$, the identity. Let F be a path homotopy between $p \circ h$ and e_{b_0}. Lift F to $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0, 0) = e_0$, a path homotopy between h (the lift of $p \circ h$) and e_{e_0} (the lift of e_{b_0}). Then $[h] = e$ in $\pi_1(E, e_0)$.

(ii) Given $f, g\colon I \to B$ two loops at b_0, we can lift them to paths $\tilde{f}, \tilde{g}\colon I \to E$ beginning at e_0. By definition, $\phi([f]) = \tilde{f}(1)$, $\phi([g]) = \tilde{g}(1)$. We want to show that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$ (meaning $[f] = [g]$ in $\pi_1(B, b_0)/H$). First suppose there is some $[h] \in H$ such that $[f] = [h * g]$. By definition, $h = p \circ \tilde{h}$ for a loop $\tilde{h}$ at e_0 in E. Then $\tilde{h} * \tilde{g}$ is a lift of $h \circ g$ starting at e_0, and similarly $\tilde{f}$ is too. Since $[f] = [h * g]$, uniqueness gives that $\tilde{f}(1) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1)$.

Conversely, let $\tilde{f}(1) = \tilde{g}(1)$, so that $\tilde{h} \coloneqq \tilde{f} * \overline{\tilde{g}}$ is a loop in E at e_0. Moreover, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$, so there is a path homotopy $F\colon I \times I \to E$ between $\tilde{h} * \tilde{g}$ and $\tilde{f}$. Then $p \circ F$ is a path homotopy in B between $h * g$ and f, where $h = p \circ \tilde{h}$, and $[h] \in H$ by definition. Thus, $[f] = [h] * [g] \in H * [g]$.

Lastly, if E is path connected, then ϕ is surjective, so the induced map is also (ϕ is just the projection map composed with Φ).

(iii) Let $g = e_{b_0}$. Then $\phi([g]) = e_0$. By (ii), $\phi([f]) = \phi([g]) = e_0$ (i.e. f is a loop) if and only if $[f] \in H * [g] = H$. $\square$

Theorem 22. Let $X = U \cup V$ where $U, V \subseteq X$ are open and $U \cap V$ is path connected. Let $x_0 \in U \cap V$ and let $i\colon U \to X$, $j\colon V \to X$ be the inclusion maps. Then the images of

$$i_*\colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad j_*\colon \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$. That is, any $[g] \in \pi_1(X, x_0)$ can be written as $[g] = [g_1] * \cdots * [g_k]$ where each $[g_l]$ lies in one of the images (is a loop in U or V at x_0).

Proof. Let $[f] \in \pi_1(X, x_0)$. Then applying the Lebesgue lemma to $\{f^{-1}(U), f^{-1}(V)\}$, we can find

$$0 = a_0 < a_1 < \cdots < a_N = 1$$

such that $f([a_{k-1}, a_k])$ is contained in U or V for each $k = 1, \ldots, N$. Now given $k = 0, \ldots, N$, check whether $f(a_k) \in U \cap V$. This is true for $k = 0, N$, since f is a loop at $x_0 \in U \cap V$. If, say, $f(a_k) \notin U \cap V$, then since $f(a_k) \in U \cup V$, we may assume that $f(a_k) \notin V$, so that

$f([a_{k-1}, a_k]), f([a_k, a_{k+1}]) \subseteq U$. Thus, we can just remove a_k. Repeating this process finitely many times, we may assume that $f(a_k) \in U \cap V$ for every $k = 0, \ldots, N$. Now for $k = 1, \ldots, N$, define

$$f_k(t) \coloneqq f((1-t)a_{k-1} + ta_k),$$

so that $f_k \colon I \to X$ is a path contained in U or V, and $f_1 * \cdots * f_N$ is a reparametrization of f. Thus,

$$[f] = [f_1] * \cdots * [f_N].$$

Now for each $k = 1, \ldots, N-1$, choose a path $\alpha_k \colon I \to U \cap V$ with $\alpha_k(0) = x_0$, $\alpha_k(1) = f(a_k)$. Let

$$\alpha_0 = \alpha_N = e_{x_0}.$$

Finally, define $g_k \coloneqq \alpha_{k-1} * f_k * \overline{\alpha_k}$, a loop in X at x_0 contained in either U or V, and

$$[g_1] * \cdots * [g_N] = [f_1] * \cdots * [f_N] = [f].$$

□

Corollary. Let $X = U \cup V$ for U, V open and simply connected. If $U \cap V$ is non-empty and path connected, then X is simply connected.

Proof. From a long time ago, know that X is path connected. Now if $x_0 \in U \cap V$, observe that $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ are trivial. Since the images of the inclusions generate $\pi_1(X, x_0)$, we also have that $\pi_1(X, x_0)$ is trivial. □

Example. We can show again that $\pi_1(\mathbb{S}^n, x)$ is trivial for $n \geq 2$. Write $\mathbb{S}^n = U \cup V$ where $U = \mathbb{S}^n \backslash \{N\}$ and $V = \mathbb{S}^n \backslash \{S\}$. Then U and V are homeomorphic to $\mathbb{R}^n$ via stereographic projection. Moreover, $U \cap V$ is homeomorphic to $\mathbb{R}^n \backslash \{0\}$ via stereographic projection, a path connected space. By the corollary, $\pi_1(\mathbb{S}^n, x)$ is trivial. ◇

Lecture 21: Algebra review

Before introducing the main theorem of this section, we need to review some algebra preliminaries (without proofs - see textbooks). Given a collection $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ of groups, there are three different products we can define.

1. Cartesian Product:
$$\prod_{\alpha\in\mathcal{A}} G_\alpha = \{(g_\alpha)_{\alpha\in\mathcal{A}} : g_\alpha \in G_\alpha \ \forall\alpha\in\mathcal{A}\},$$
where the group operation is component-wise products.

2. Direct Product:
$$\bigoplus_{\alpha\in\mathcal{A}} G_\alpha = \{(g_\alpha)_{\alpha\in\mathcal{A}} : g_\alpha \in G_\alpha \ \forall\alpha\in\mathcal{A}, \text{ and } g_\alpha = \mathrm{id}_\alpha \text{ for all but finitely many } \alpha\},$$
where the group operation is component-wise products.

Remark. If $\mathcal{A}$ is finite, the Cartesian and direct products agree, but they differ for infinite $\mathcal{A}$.

There is an issue with the above two products. In both, different factors G_α, G_β ($\alpha \neq \beta$) always commute! Thus, need the third type of product, the *free product*:

3. Free Product:
$$\mathop{\ast}_{\alpha\in\mathcal{A}} G_\alpha = \{g_1 \dots g_m : g_1 \dots g_m \text{ is a reduced word of finite length } m \in \mathbb{N}\}.$$

Here, we define a *reduced word* to be $g_1 \dots g_m$ where for every $1 \leq i \leq m$ there is $\alpha_i \in \mathcal{A}$ such that $g_i \in G_{\alpha_i}\backslash\{e_{\alpha_i}\}$ and $\alpha_i \neq \alpha_{i+1}$ for each i (i.e. g_i and g_{i+1} are in different groups). For $m = 0$, we have the *empty word*, which is the identity element. Also note that any unreduced word can be *simplified* by writing adjacent letters in the same G_α as one letter, and removing the identity element. For the free product to be a group, we need to define a group operation. We will take the group operation to be juxtaposition then applying simplification: for $g_1 \dots g_m$, $h_1 \dots h_n$ in $\ast_{\alpha\in\mathcal{A}} G_\alpha$, define
$$(g_1 \dots g_n) \cdot (h_1 \dots h_n) \coloneqq g_1 \dots g_n h_1 \dots h_n$$
where we simplify after taking juxtaposing.

Theorem 23. The set $\ast_{\alpha\in\mathcal{A}} G_\alpha$ with the juxtaposition plus simplification operation is a group.

Proof. The only non-trivial property to check is associativity. Let W be the set of all reduced words of finite length. Given $g \in G_\alpha$, $\alpha \in \mathcal{A}$, define a map
$$\begin{aligned} L_g \colon W &\longrightarrow W \\ g_1 \dots g_m &\longmapsto \text{simplification of } g g_1 \dots g_n. \end{aligned}$$

Then $L_g \circ L_{g'} = L_{gg'}$ and L_{e_α} is the identity, where $g, g' \in G_\alpha$ and e_α is the identity in G_α. Thus, L_g is invertible with inverse $L_{g^{-1}}$. Hence, if $P(W)$ is the set of permutations of W, then $L_g \in P(W)$ and $P(W)$ is a group. Moreover, $G_\alpha \to P(W)$ where $g \mapsto L_g$ is a group homomorphism. Now define
$$L \colon W \longrightarrow P(W)$$

$$g_1 \dots g_n \longmapsto L_{g_1} \circ \dots \circ L_{g_n},$$

an injective map of sets, since $(L_{g_1} \circ \dots \circ L_{g_n})(\varnothing) = g_1 \dots g_n$. Thus, $L(W) \subseteq P(W)$ and the product on W (juxtaposition plus simplification) via L becomes composition. But composition is associative in $P(W)$, so the product in W must be too. ☐

Example. Let $G_\alpha = \mathbb{Z}$ for all $\alpha \in \mathcal{A}$. Then $\ast_{\alpha\in\mathcal{A}} \mathbb{Z}$ is called the free group with $\mathcal{A}$ generators. ◇

Example. $\mathbb{Z} * \mathbb{Z} = \{$reduced words in a, b and their powers$\}$. e.g. $a^5b^2a^{-1}b^9ab^{-2}$. ◇

Example. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \{$reduced words in a, b without their powers$\}$. e.g. ab, $bababa$. ◇

Given a collection of groups $\{G_\alpha\}_{\alpha\in\mathcal{A}}$, there is a natural inclusion map $\iota_\alpha \colon G_\alpha \to \ast_{\alpha\in\mathcal{A}} G_\alpha$, where $g \neq e_\alpha$ is mapped to the word "g" and e_α is mapped to the empty word. Using ι_α, we often identify G_α with $\iota_\alpha(G_\alpha)$.

Theorem 24 (Universal Property for Free Groups)**.** Given a collection of groups $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ and group homomorphisms $\varphi_\alpha \colon G_\alpha \to H$, there is a unique homomorphism $\varphi\colon \ast_{\alpha\in\mathcal{A}} G_\alpha \to H$ such that $\varphi \circ \iota_\alpha = \varphi_\alpha$ for every $\alpha \in \mathcal{A}$.

Proof. Uniqueness: If φ exists, then given $g_i \in G_{\alpha_i}$, we have

$$\varphi(g_1 \dots g_n) = \varphi(g_1) \dots \varphi(g_n) = \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n),$$

which is uniquely determined.

Existence: Define φ by $\varphi(\varnothing) = e_H$ and $\varphi(g_1 \dots g_n) = \varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n)$. Then it is quick check to see that φ is a group homomorphism. ☐

Example. Let G_1, G_2 be two groups and $\varphi_i \colon G_i \to G_1 \times G_2$ be the natural inclusions, $i = 1, 2$. Then the proposition above gives a homomorphism $\varphi\colon G_1 * G_2 \to G_1 \times G_2$ that is surjective. To see this, let $(g, h) \in G_1 \times G_2$ be given. Then

$$\begin{cases} (g, e) = \varphi_1(g) = \varphi(\iota_1(g)), \\ (e, h) = \varphi_2(h) = \varphi(\iota_2(h)), \end{cases}$$

so that

$$(g, h) = (g, e)(e, h) = \varphi(\iota_1(g)\iota_2(h)).$$

◇

Before moving back into the realm of topology, we must recall that a *normal subgroup* $H \subseteq G$ of a group G is a subgroup satisfying $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$. We often write $H \trianglelefteq G$ to mean H is a normal subgroup of G. Given a subset $S \subseteq G$, we call the set

$$N := \bigcap_{\substack{H \trianglelefteq G \\ S \subseteq H}} H$$

the normal subgroup *generated by S*.

Lecture 22: Seifert-Van Kampen Theorem

Now that we have an understanding of the important algebra preliminaries, let us resume our discussion of algebraic topology. Let X be a topological space and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of X with $x_0 \in \bigcap_{\alpha\in\mathcal{A}} U_\alpha$ and $U_\alpha \cap U_\beta$ is path connected for all $\alpha, \beta \in \mathcal{A}$. We have the inclusions $\iota_\alpha \colon U_\alpha \to X$, and the induced homomorphisms

$$j_\alpha \coloneqq (\iota_\alpha)_* \colon \pi_1(U_\alpha, x_0) \to \pi_1(X, x).$$

By the universal property for free groups, there exists a homomorphism

$$\Phi \colon \mathop{\ast}_{\alpha\in\mathcal{A}} \pi_1(U_\alpha, x_0) \to \pi_1(X, x_0)$$

which extends the j_α. The same proof as in the case of two U_α shows that the images of all j_α together generate $\pi_1(X, x_0)$. Thus, given $[f] \in \pi_1(X, x_0)$, we can write $[f] = [g_1] * \ldots * [g_N]$ where for any $1 \le i \le N$ there is $\alpha(i) \in \mathcal{A}$ such that $[g_i] \in \operatorname{im}(j_{\alpha(i)})$. Hence, $[g_i] \in \operatorname{im}(\Phi)$ for all $1 \le i \le N$. But Φ is a group homomorphism, so $[f] \in \operatorname{im}(\Phi)$ – Φ is surjective!

We note that Φ is, in general, not injective: if $\iota_{\alpha\beta} \colon \pi_1(U_\alpha \cap U_\beta, x_0) \to \pi_1(U_\alpha, x_0)$ is the homomorphism induced by the inclusion $U_\alpha \cap U_\beta \hookrightarrow U_\alpha$, then the diagram

$$\begin{array}{ccccc} & & U_\alpha \cap U_\beta & & \\ & \swarrow & & \searrow & \\ U_\alpha & & & & U_\beta \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

commutes. Now by functoriality, $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$, so for any $[\gamma] \in \pi_1(U_\alpha \cap U_\beta, x)$,

$$\Phi(i_{\alpha\beta}([\gamma])) = \Phi(i_{\beta\alpha}([\gamma])),$$

or $i_{\alpha\beta}([\gamma])(i_{\beta\alpha}([\gamma]))^{-1} \in \ker(\Phi)$.

Theorem 25 (Seifert-Van Kampen)**.** Let X be a topological space, $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of X with $x_0 \in \bigcap_{\alpha\in\mathcal{A}} U_\alpha$, $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$ path connected for all $\alpha, \beta, \gamma \in \mathcal{A}$. Then the kernel of $\Phi \colon \ast_{\alpha\in\mathcal{A}} \pi_1(U_\alpha, x_0) \to \pi_1(X, x_0)$ is the normal subgroup N generated by

$$\{i_{\alpha\beta}([\gamma])(i_{\beta\alpha}([\gamma])^{-1} : [\gamma] \in \pi_1(U_\alpha \cap U_\beta, x_0),\ \alpha \ne \beta\}.$$

In particular, Φ induces a group isomorphism

$$\pi_1(X, x_0) \cong \left(\mathop{\ast}_{\alpha\in\mathcal{A}} \pi_1(U_\alpha, x_0) \right) / N.$$

The proof is long, so is omitted here.

Corollary. Let $X = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ for some $U_\alpha \subseteq X$ open, $U_\alpha \cap U_\beta \cap U_\gamma$ path connected for all $\alpha, \beta, \gamma \in \mathcal{A}$, and let $x_0 \in \bigcap_{\alpha\in\mathcal{A}} U_\alpha$. If $U_\alpha \cap U_\beta$ is simply connected for every $\alpha \ne \beta$, then

$$\pi_1(X, x_0) \cong \mathop{\ast}_{\alpha\in\mathcal{A}} \pi_1(U_\alpha, x_0).$$

Indeed, for $U_\alpha \cap U_\beta$ simply connected, the maps $i_{\alpha\beta}$ and $i_{\beta\alpha}$ are immediately trivial, so N is trivial.

Corollary. Let $\{(X_\alpha, x_\alpha)\}_{\alpha\in\mathcal{A}}$ be pointed spaces such that for any $\alpha \in \mathcal{A}$ there is an open set $W_\alpha \subseteq X_\alpha$ that contains x_α and x_α is a deformation retract of W_α. Then

$$\pi_1(\bigvee_{\alpha\in\mathcal{A}} X_\alpha, x_0) \cong \mathop{\ast}_{\alpha\in\mathcal{A}} \pi_1(X_\alpha, x_\alpha),$$

where x_0 is the image of the x_α.

Proof. For each $\alpha \in \mathcal{A}$, define $U_\alpha = X_\alpha \vee \bigvee_{\beta\neq\alpha} W_\beta$, which is an open subset of $\bigvee_{\gamma\in\mathcal{A}} X_\gamma$. Then $X_\alpha \subseteq U_\alpha$, and since W_β deformation retracts onto x_β, we have that X_α is a deformation retract of U_α. Thus, $\pi_1(U_\alpha, x_0) \cong \pi_1(X_\alpha, x_\alpha)$. Moreover, the intersection of two or more U_α is equal to $\bigvee_{\gamma\in\mathcal{A}} W_\gamma$, which deformation retracts onto $\{x_0\}$, hence these are simply connected. By the above corollary, we have

$$\pi_1(\bigvee_{\alpha\in\mathcal{A}} X_\alpha, x_0) \cong \mathop{\ast}_{\alpha\in\mathcal{A}} \pi_1(X_\alpha, x_\alpha).$$

□

Example. Let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ and let $x_0 \in \mathbb{S}^1$ be a basepoint. Then given $y \in \mathbb{S}^1$, $y \neq x_0$, we have $S^1\backslash\{y\}$ is homeomorphic to $\mathbb{R}$ and contains x_0. Let $\Phi\colon \mathbb{S}^1\backslash\{y\} \to \mathbb{R}$ be a homeomorphism. Then

$$\Phi^{-1}((\Phi(x_0) - 1, \Phi(x_0) + 1))$$

is an open neighbourhood of x_0 in $\mathbb{S}^1$, which deformation retracts onto x_0. By the above corollary,

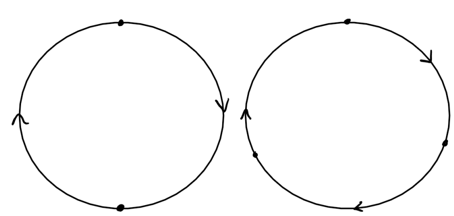
$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1, x_0) \cong \mathbb{Z} \times \mathbb{Z}.$$

Extending this argument gives, more generally,

$$\pi_1(\bigvee_{\alpha\in\mathcal{A}} \mathbb{S}^1, x_0) \cong \mathop{\ast}_{\alpha\in\mathcal{A}} \mathbb{Z}.$$

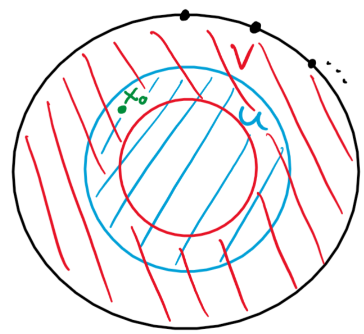
◇

Example (n-fold dunce cap)**.** For $n \geq 2$, define a space X by taking $\overline{B}^2 \subseteq \mathbb{R}^2$ and subdividing $\mathbb{S}^1 = \partial\overline{B}^2$ into n equal arcs and identifying them all (respecting orientation): ◇



For $n \geq 3$, one can also think of these as polygons.

Fix a basepoint $x_0 \in X$, the image of some point in B^2, and let U, V be defined as:



Then we see that $X = U \cup V$, U is convex hence contractible, V deformation retracts onto $q(\mathbb{S}^1)$ where $q\colon \overline{B}^2 \to X$ is the usual quotient map, so V is homotopy equivalent to $\mathbb{S}^1$. Moreover, $U \cap V$ is also homotopy equivalent to $\mathbb{S}^1$.

Let γ be a loop in $U \cap V$, which is homotopic to the loop in $\mathbb{S}^1$ traversing $\mathbb{S}^1$ once clockwise. Let σ be the composition of q with one arc on the partitioned $\mathbb{S}^1$. Then $[\sigma]$ generates $\pi_1(V, x_0) \cong \mathbb{Z}$ and $[\gamma]$ generated $\pi_1(U \cap V, x_0) \cong \mathbb{Z}$. Let

$$i\colon \pi_1(U \cap V, x_0) \to \pi_1(V, x_0), \quad j\colon \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$$

be the group homomorphisms induced by the inclusion maps. Then j is trivial, while $i([\gamma])$ is the homotopy class of γ inside V, i.e. the class of the image of the loop that traverses $\mathbb{S}^1$ once clockwise. In V, this is equal to

$$\underbrace{[\sigma] * \cdots * [\sigma]}_{n \text{ times}} = n[\sigma].$$

Thus, $i([\gamma])(j([\gamma])^{-1} = n[\sigma]$. Hence, $N \subseteq \pi_1(U, x_0) * \pi_1(V, x_0)$ is the normal subgroup generated by $n[\sigma]$. By Seifert-van Kampen, we see that

$$\pi_1(X, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0))/N \cong \mathbb{Z}/n\mathbb{Z}.$$

Remark (Fundamental Group of the Projective Plane)**.** $\mathbb{RP}^2$ is homeomorphic to the 2-fold dunce cap, so $\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. Indeed,

$$\mathbb{RP}^2 = (\mathbb{R}^2 \backslash \{0\})/(p \sim \lambda p) = \mathbb{S}^2/(p \sim -p),$$

where $\lambda \in \mathbb{R}\backslash\{0\}$. Now, $\mathbb{S}^2/(p \sim -p)$ is equivalent the upper hemisphere with no identification plus the equator with the $p \sim -p$ identification, so up to homeomorphism this is the 2-fold dunce cap.

Lecture 23: Covering Spaces ctd.

We resume our study of covering spaces.

Theorem 26 (Lifting Theorem)**.** Let $p\colon E \to B$ be a covering space with $p(e_0) = b_0$ and X a path connected and locally path connected space. Suppose $f\colon X \to B$ is a continuous map with $f(x_0) = b_0$. Then there exists a continuous map $\tilde{f}\colon X \to E$ such that $f = p \circ \tilde{f}$ (i.e. f lifts to $\tilde{f}$) and $\tilde{f}(x_0) = e_0$. if and only if $f_*\pi_1(X, x_0) \subseteq p_*\pi_1(E, e_0)$. If this holds, then $\tilde{f}$ is unique.

Proof. First suppose that such a $\tilde{f}$ exists. Since $f = p \circ \tilde{f}$, functoriality gives that $f_* = p_* \circ \tilde{f}_*$. Thus,

$$f_*\pi_1(X, x_0) = p_*(\tilde{f}_*\pi_1(X, x_0)) \subseteq p_*\pi_1(E, e_0).$$

For uniqueness, fix $x \in X$ and choose a path $\alpha\colon I \to X$ with $\alpha(0) = x_0$, $\alpha(1) = x$. Then $f \circ \alpha\colon I \to B$ is a path starting at b_0. Let $\gamma\colon I \to E$ be its lift starting at e_0, i.e. $p \circ \gamma = f \circ \alpha$. Then $\tilde{f} \circ \alpha$ is also a lift of $f \circ \alpha$ starting at e_0, so by uniqueness we have $\gamma = \tilde{f} \circ \alpha$. Hence, $\tilde{f}(x) = \tilde{f}(\alpha(1)) = \gamma(1)$, which is uniquely determined.

Conversely, let $x \in X$ be given and choose a path $\alpha\colon I \to X$ with $\alpha(0) = x_0$, $\alpha(1) = x$. Suppose $\gamma\colon I \to E$ is the lift of $f \circ \alpha$ starting at e_0 and define $\tilde{f}(x) = \gamma(1)$. We need to show that $\tilde{f}$ is well-defined and continuous.

Let $\beta\colon I \to X$ be another path from x_0 to x and $\overline{\beta}$ its reverse path. Lift $f \circ \overline{\beta}\colon I \to B$ to a path $\delta\colon I \to E$ with $\delta(0) = \gamma(1)$, so that $\delta * \gamma$ makes sense and is a lift of

$$(p \circ \gamma) * (p \circ \delta) = (f \circ \alpha) * (f \circ \overline{\beta}) = f \circ (\alpha * \overline{\beta}).$$

Thus, $[f \circ (\alpha * \overline{\beta})] \in \operatorname{im}(p_*)$. We proved before that for such a loop, the lift (which is $\gamma * \delta$) is a loop in E. Thus, $\delta(1) = \gamma(0) = e_0$. But then $\overline{\delta}$, which is a lift of $f \circ \beta$, starts at e_0 and ends at $\overline{\delta}(1) = \gamma(1)$. Hence, $\tilde{f}(x) = \overline{\delta}(1)$, so indeed $\tilde{f}$ is well-defined.

We show $\tilde{f}$ is continuous. Let $V \subseteq E$ be an open set containing $\tilde{f}(x)$. We find an open set $W \subseteq X$ so that $\tilde{f}(W) \subseteq V$. Let $U \subseteq B$ be an evenly covered open set containing $f(x)$, $p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$ for some collection of open disjoint $V_\alpha \subseteq E$ with $p|_{V_\alpha}$ a homeomorphism. Let us take $\tilde{f}(x) \in V_0$, and up to shrinking U, we may assume $V_0 \subseteq V$. Since f is continuous, $x \in f^{-1}(U) \subseteq X$ is open, so by local path connectedness there is path connected open set $W \subseteq f^{-1}(U)$ containing x. Let us show that $\tilde{f}(W) \subseteq V_0$. Given $y \in W$, take $\beta\colon I \to W$ a path starting at x and ending at y. Then $\tilde{f}(y)$ is exactly the endpoint of the lift of $f \circ (\alpha * \beta)$ starting at e_0. Suppose γ is the lift of $f \circ \alpha$, as before, then $(f \circ \beta)(I) \subseteq f(W) \subseteq U$, so $\delta \coloneqq (p|_{V_0})^{-1} \circ f \circ \beta$ is a lift of $f \circ \beta$ starting at $\tilde{f}(x)$., $\delta(I) \subseteq V_0$. Thus, $\gamma * \delta$ is the lift of $f \circ (\alpha * \beta)$ starting at e_0 and ending at $\delta(1) \in V_0$. Thus, $\tilde{f}(y) \in V_0$, as desired. □

Given $p\colon E \to B$ a covering space, we call another covering space $p'\colon E' \to B$ equivalent to p if there is a homeomorphism $h\colon E \to E'$ such that $p = p' \circ h$. Being equivalent is an equivalence relation on covering spaces.

Example. Define a map $\mathbb{S}^1 \to \mathbb{S}^1$ by $z \mapsto z^n$. Then in general this is not an equivalence of $\mathbb{S}^1$. ⋄

Theorem 27. Let $p\colon E \to B$, $p'\colon E' \to B$ be covering spaces with $p(e_0) = b_0 = p(e_0')$, where E, E' are path connected and locally path connected. Then there is an equivalence $h\colon E \to E'$ with

$h(e_0) = e'_0$ if and only if

$$H \coloneqq p_* \pi_1(E, e_0), \quad H' \coloneqq p'_* \pi_1(E', e'_0)$$

are equal ($H = H'$). If such an h exists, then it is unique.

Proof. By the topological invariance of π_1, $h_* \pi_1(E, e_0) = \pi_1(E', e'_0)$, so

$$H = p'_* h_* \pi_1(E, e_0) = p'_* \pi_1(E'_0, e'_0) = H'.$$

Conversely, if $p_* \pi_1(E, e_0) = p'_* \pi_1(E', e'_0)$, then by the lifting theorem there is a lift $h \colon E \to E'$ with $h(e_0) = e'_0$ and $p' \circ h = p$. Similarly, there is a lift $k \colon E' \to E$ with $k(e'_0) = e_0$ and $p \circ k = p'$. Then $k \circ h \colon E \to E$ satisfies

$$p \circ k \circ h = p' \circ h = p,$$

so $k \circ h$ is a lifting of p and $(k \circ h)(e_0) = e_0$. By uniqueness of liftings, $k \circ h = \mathrm{id}_E$, and similarly we have $h \circ k = \mathrm{id}_{E'}$. Thus, h is an equivalence. Uniqueness is an immediate consequence of uniqueness in the lifting theorem. □

What if we discard basepoints? Let G be a group, and H_1, H_2 two subgroups of G. We say H_1, H_2 are conjugate if there is $g \in G$ so that $gH_1g^{-1} = H_2$. This is an equivalence relation.

Lemma 10. Let $p \colon E \to B$ be a covering space with $p(e_0) = b_0 = p(e_1)$, where E is path connected and locally path connected. Let

$$H_0 = p_* \pi_1(E, e_0), \quad H_1 = p_* \pi_1(E, e_1)$$

(these are subgroups of $\pi_1(B, b_0)$).

(a) H_0 and H_1 are conjugate. More precisely, if $\gamma \colon I \to E$ is a loop beginning at e_0 and ending at e_1 and $\alpha \coloneqq p \circ \gamma$ is a loop in B at b_0, then $[\alpha] * H_1 * [\overline{\alpha}] = H_0$.

(b) Given $H \subseteq \pi_1(B, b_0)$ a subgroup conjugate to H_0, then there exists $e_2 \in p^{-1}(b_0)$ such that $p_* \pi_1(E, e_2) = H$.

Proof. (a) Given $[h] \in H_1$, we have $[h] = p_*[\tilde{h}]$ for some loop $\tilde{h}$ at e_1. Let $\tilde{k} = \gamma * \tilde{h} * \overline{\gamma}$, which is a loop in E at e_0. Then

$$p_*[\tilde{k}] = [p \circ \gamma] * [p \circ \tilde{k}] * [p \circ \overline{\gamma}] = [\alpha] * [h] * [\overline{\alpha}] \in H_0.$$

Thus, $[\alpha] * H_1 * [\overline{\alpha}] \subseteq H_0$. Reversing the roles of e_0 and e_1 gives the other direction.

(b) By assumption, there is a loop $\alpha \colon I \to B$ a loop at b_0 such that $H_0 = [\alpha] * H * [\overline{\alpha}]$. Let γ be the lift of α to a path in E starting at e_0 and let $e_2 \coloneqq \gamma(1)$. By (a),

$$[\alpha] * p_* \pi_1(E, e_2) * [\overline{\alpha}] = H_0 = [\alpha] * H * [\overline{\alpha}],$$

so the claim holds. □

Theorem 28. Let $p \colon E \to B$, $p' \colon E' \to B$ be two covering spaces with $p(e_0) = b_0 = p'(e'_0)$, where E, E' are path connected and locally path connected. Then p is equivalent to p' if and only if $p_* \pi_1(E, e_0)$ and $p'_* \pi_1(E', e'_0)$ are conjugate subgroups of $\pi_1(B, b_0)$.

Proof. Let $h\colon E \to E'$ be an equivalence and let $e_1' := h(e_0)$. Let

$$H_0 = p_*\pi_1(E, e_0), \quad H_0' = p'_*\pi_1(E', e_0'), \quad H_1' = p'_*\pi_1(E', e_1').$$

By the above theorem, $H_0 = H_1'$, and by the lemma, H_0' is conjugate to H_1'.

Conversely, if H_0 and H_0' are conjugate, then the lemma gives $e_1 \in E$ with $p_*\pi_1(E, e_1) = H_0'$ and $p(e_1) = b_0$. By the theorem, p and p' are equivalent. □

Example. Let $B = \mathbb{S}^1$. Then $\pi_1(B, b_0) \cong \mathbb{Z}$ is abelian, so any two conjugate subgroups are equal. In particular, any two coverings of $\mathbb{S}^1$ are equivalent if and only if their corresponding subgroups are equal. ◇

Recall that any subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some positive integer n. Thus, every covering space $p\colon E \to \mathbb{S}^1$ with E path connected and locally path connected corresponds to $n\mathbb{Z}$ for some $n \in \mathbb{N}$.

Lectures 24 & 25: Existence of covering spaces

Given a pointed space (B, b_0) and $H \subseteq \pi_1(B, b_0)$ a subgroup, we wish to determine if a covering space $p\colon E \to B$ with E path connected and locally path connected such that $p_*\pi_1(E, e_0) = H$ exists. If such a map exists, then B is path connected and locally path connected. Moreover, if H is trivial, since p_* is injective, we may assume E is simply connected.

Lemma 11. Let $p\colon E \to B$ be a covering space with E simply connected. Then for any $b \in B$ there is an open set $U \subseteq B$ containing b such that

$$\iota_*\colon \pi_1(U, b) \longrightarrow \pi_1(B, b)$$

is the trivial homomorphism (i.e. any loop in U at b is path-homotopic to e_b inside B).

Proof. Let $b \in U \subseteq B$ be an open set evenly covered by p, with $p^{-1}(U) = \coprod_{\alpha \in \mathcal{A}} V_\alpha$. Choose $\alpha \in \mathcal{A}$ and $e_0 \in V_\alpha$ with $p(e) = b$. Given a loop f in U at b, $\tilde{f} := (p|_{V_\alpha})^{-1} \circ f$ is a loop in V_α based at e. But E is simply connected, so there is $F\colon I \times I \to E$ a path homotopy between $\tilde{f}$ and e_e. Then $p \circ F$ is a path homotopy between $f = p \circ \tilde{f}$ and e_b. □

We say that a space B is *semilocally simply connected* if for any $b \in B$ there is an open set $U \subseteq B$ containing b such that

$$\iota_*\colon \pi_1(U, b) \longrightarrow \pi_1(B, b)$$

is the trivial homomorphism.

Theorem 29 (Classification of covering spaces)**.** Let B be path connected, locally path connected, and semilocally simply connected. Fix $b_0 \in B$ and let $H \subseteq \pi_1(B, b_0)$ be a subgroup. Then there exists a covering space $p\colon E \to B$ with E path connected and locally path connected, $e_0 \in E$ with $p(e_0) = b_0$, and $p_*\pi_1(E, e_0) = H$. Moreover, (E, e_0) is unique up to base point preserving equivalence.

Proof. Define

$$\mathcal{P} := \{\text{paths } \gamma\colon I \to B \text{ with } \gamma(0) = b_0\}.$$

For $\alpha, \beta \in \mathcal{P}$, define an equivalence relation by $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha * \overline{\beta}] \in H$. Denote by $\alpha^\#$ the equivalence class of $\alpha \in \mathcal{P}$ and define $E := \mathcal{P}/\sim$ and $p\colon E \to B$ with $\alpha^\# \mapsto \alpha(1)$. Then p is a well-defined mapping of sets. Morever, p is surjective, since B is path connected.

Observation 1: if $\alpha, \beta \in \mathcal{P}$ have $\alpha(1) = \beta(1)$, and $\alpha \simeq_p \beta$, then $\alpha * \overline{\beta} \simeq_p e_{b_0}$. Hence $[\alpha * \overline{\beta}] = \mathrm{id} \in H$, and so $\alpha^\# = \beta^\#$.

Observation 2: Let $\alpha, \beta \in \mathcal{P}$ satisfy $\alpha^\# = \beta^\#$. Then given any path $\delta\colon I \to B$ with $\delta(0) = \alpha(1)$, we have $\alpha * \delta$ and $\beta * \delta$ end at the same end point and

$$[\alpha * \delta * \overline{(\beta * \delta)}] = [\alpha * \overline{\beta}] \in H.$$

Hence $(\alpha * \delta)^\# = (\beta * \delta)^\#$.

We define a topology on E. Given $\alpha \in \mathcal{P}$, choose an open and path connected set $U \subseteq B$ containing $\alpha(1)$. Define

$$B(U, \alpha) := \{(\alpha * \delta)^\# : \delta\colon I \to U \text{ a path, } \delta(0) = \alpha(1)\}.$$

Notice that $B(U, \alpha)$ is non-empty, as $\alpha^\# \in B(U, \alpha)$ since $\alpha^\# = (\alpha * e_{\alpha(1)})^\#$.

Claim 1: if $\beta \in \mathcal{P}$ has $\beta^\# \in B(U,\alpha)$, then $\alpha^\# \in B(U,\beta)$ and $B(U,\alpha) = B(U,\beta)$.

To see this, notice that $\beta^\# = (\alpha * \delta)^\#$ for some $\delta\colon I \to U$ with $\delta(0) = \alpha(1)$, $\overline{\delta}(0) = \delta(1) = \beta(1)$, so Observation 2 gives

$$(\beta * \overline{\delta})^\# = (\alpha * \delta * \overline{\delta})^\# = \alpha^\#.$$

Hence $\alpha^\# \in B(U,\beta)$. Given any $(\beta * \gamma)^\# \in B(U,\beta)$, where $\gamma\colon I \to U$ is a path with $\gamma(0) = \beta(1)$, we have

$$(\beta * \gamma)^\# = (\alpha * (\delta * \gamma))^\# \in B(U,\alpha).$$

Thus $B(U,\beta) \subseteq B(U,\alpha)$, and symmetry gives the reverse inclusion.

Claim 2: we check that $\{B(U,\alpha)\}$ forms a basis for the topology on E.

It is straightforward to see that they cover E, so we need only check that given any two $B(U_1,\alpha_1)$, $B(U_2,\alpha_2)$ with non-empty intersection, there exists $B(V,\beta) \subseteq B(U_1,\alpha_1) \cap B(U_2,\alpha_2)$. To see this, given $\beta^\#$ in the intersection, Claim 1 gives that $B(U_i,\alpha_1) = B(U_i,\beta)$ for $i = 1,2$. Now choose an open and path connected set $V \subseteq U_1 \cap U_2$ containing $\beta(1)$. Then by definition, $B(V,\beta) \subseteq B(U_i,\beta)$ for $i = 1,2$, as desired. Hence, there is a unique topology on E for which the $B(U,\alpha)$ form a basis.

We now show that p is an open map. In particular, we show that for any $B(U,\alpha)$ with U open and path connected and $\alpha(1) \in U$, that $p(B(U,\alpha))$ is open in B. Given some $x \in U$, choose a path $\delta\colon I \to U$ with $\delta(0) = \alpha(1)$ and $\delta(1) = x$. Then $(\alpha * \delta)^\# \in B(U,\alpha)$ and

$$p((\alpha * \delta)^\#) = (\alpha * \delta)(1) = \delta(1) = x.$$

Thus, $U \subseteq p(B(U,\alpha))$. Conversely, $p(B(U,\alpha)) \subseteq U$, as given $(\alpha * \delta)^\#$, we have

$$p((\alpha * \delta)^\#) = \delta(1) \in U.$$

Thus, p is an open map. This seems useless right now, but will be used shortly.

We now show p is continuous. Given any open set $W \subseteq B$ open, we want to show $p^{-1}(W) \subseteq E$ is open. Let $\alpha^\# \in p^{-1}(W)$ be given, so that $p(\alpha^\#) = \alpha(1) \in W$. By local path connectedness of B, there is some open and path connected set U so that $\alpha(1) \in U$ and $U \subseteq W$. Then $\alpha^\# \in B(U,\alpha) \subseteq E$ (using the constant path) and $p(B(U,\alpha)) = U \subseteq W$, so $B(U,\alpha) \subseteq p^{-1}(W)$.

We now have p is well-defined, continuous, and surjective. We show p is a covering space. Given $b \in B$, we find an open set U containing b that is evenly covered. Since B is semilocally simply connected, there is an open set U containing b with U path connected (up to replacing U with the path component) and

$$\iota_*\colon \pi_1(U,b) \longrightarrow \pi_1(B,b)$$

is the trivial homomorphism.

Claim 3: we have $p^{-1}(U) = \bigcup_\alpha B(U,\alpha)$ over all paths $\alpha\colon I \to B$ with $\alpha(0) = b_0$ and $\alpha(1) = b$. Note we are using the U from above here.

Given any such path α, we already have $p(B(U,\alpha)) = U$, so

$$B(U,\alpha) \subseteq p^{-1}(p(B(U,\alpha))) = p^{-1}(U).$$

Taking the union over all such α, we have $\bigcup_\alpha B_{U,\alpha} \subseteq p^{-1}(U)$. Now given $\beta^\# \in p^{-1}(U)$, we have $\beta(1) = p(\beta^\#) \in U$. Choose $\delta\colon I \to U$ with $\delta(0) = b$ and $\delta(1) = \beta(1)$. Define $\alpha \coloneqq \beta * \overline{\delta}$. Then $\alpha(0) = b_0$ and $\alpha(1) = b$, and

$$[\alpha * \delta] = [\beta * \overline{\delta} * \delta] = [\beta],$$

so by Observation 1, $\beta^\# = (\alpha * \delta)^\#$. Hence, $\beta^\# \in \bigcup_\alpha B(U, \alpha)$.

Notice that $\bigcup_\alpha B(U, \alpha)$ (over all such α) is disjoint, as if $\beta^\# \in B(U, \alpha_1) \cap B(U, \alpha_2)$, then Claim 1 gives that

$$B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2).$$

Hence $p^{-1}(U) = \coprod_\alpha B(U, \alpha)$.

We show now that $p\colon B(U, \alpha) \to U$ is a homeomorphism, so that p is indeed a covering space. We do know that this map is continuous, open, and surjective. It remains to show injectivity. Suppose $p((\alpha * \delta_1)^\#) = p((\alpha * \delta_2)^\#)$, so $\delta_1(1) = \delta_2(1)$, where $\delta_i\colon I \to U$ satisfy $\delta_i(0) = \alpha(1) = b$ and $\delta_1(1) = \delta_2(1)$. In particular, $\delta_1 * \overline{\delta_2}$ is a loop in U at b, so $[\delta_1 * \overline{\delta_2}] \in \pi_1(U, b)$. But $\pi_1(U, b) \to \pi_1(B, b)$ is trivial by semilocal simply connectedness. Thus, $\delta_1 * \overline{\delta_2} \simeq_p e_b$ in B, so $[\alpha * \delta_1] = [\alpha * \delta_2]$, and hence $(\alpha * \delta_1)^\# = (\alpha * \delta_2)^\#$.

Now observe that E is locally path connected, since B is locally path connected and p is a local homeomorphism. It is trickier to show that E is path connected. Since $e_{b_0} \in \mathcal{P}$, we have $e_0 \coloneqq (e_{b_0})^\# \in E$, and $p(e_0) = e_{b_0}(1) = b_0$. Given a path $\alpha\colon I \to B$ with $\alpha(0) = b_0$ and a constant $c \in [0, 1]$,

$$\begin{aligned} \alpha_c\colon I &\longrightarrow B, \\ t &\longmapsto \alpha(ct) \end{aligned}$$

which is a path with $\alpha_c(0) = \alpha(0)$, $\alpha_c(1) = \alpha(c)$. Note that $\alpha_0 = e_{b_0}$ and $\alpha_1 = \alpha$. Now let

$$\begin{aligned} \tilde{\alpha}\colon I &\longrightarrow E. \\ c &\longmapsto (\alpha_c)^\# \end{aligned}$$

Then $\tilde{\alpha}$ is a map of sets, and

$$\begin{cases} \tilde{\alpha}(0) - (\alpha_0)^\# = (e_{b_0})^\# = e_0, \\ \tilde{\alpha}(1) = (\alpha_1)^\# = \alpha^\#. \end{cases}$$

Moreover, $p \circ \tilde{\alpha} = \alpha$, since

$$(p \circ \tilde{\alpha})(c) = p(\tilde{\alpha}(c)) = p((\alpha_c)^\#) = \alpha_c(1) = \alpha(c)$$

for every $c \in I$. We need to show $\tilde{\alpha}$ is continuous. Given $0 \le c < d \le 1$, let $\delta_{c,d} = \alpha|_{[c,d]}$ reparametrized to I. Then $\alpha_d = \delta_{0,d}$ and $\alpha_c * \delta_{c,d}$ differ only by reparametrization. Thus, $\alpha_d \simeq_p \alpha_c * \delta_{c,d}$. Given $c \in I$, we show $\tilde{\alpha}$ is continuous at c. Take $\alpha(c) \in U \subseteq B$ for an open and path connected set U, and define $W = B(U, \alpha_c)$ (which is open in E). Observe that $\tilde{\alpha}(c) \in W$. Since α is continuous, there is $\epsilon > 0$ such that if $|t - c| < \epsilon$, then $\alpha(t) \in U$. We need to show that if $d \in I$ satisfies $|d - c| < \epsilon$ then $\tilde{\alpha}(d) \in W$, which shows that $\tilde{\alpha}$ is continuous at c. If $d = c$ we are done. If $d > c$, call $\delta = \delta_{c,d}$, so

$$\tilde{\alpha}(d) = (\alpha_d)^\# = (\alpha_c * \delta)^\#,$$

by Observation 1. Since $\delta = \alpha|_{[c,d]}$, we then have $\delta(I) \subseteq U$. Hence $\tilde{\alpha}(d) \in B(U, \alpha_c)$. A similar argument works for the $d < c$ case. Thus, $\tilde{\alpha}$ is a path from e_0 to $\alpha^{\#}$. Since everything is connected to e_0, E is path connected.

To conclude, we need to check that $p_*\pi_1(E, e_0) = H$. Let $\alpha\colon I \to B$ be a loop in B at b_0 and $\tilde{\alpha}\colon I \to E$ its lift startin at e_0, where we use the lifting properties of covering spaces (note uniqueness). Then $[\alpha] \in p_*\pi_1(E, e_0)$ if and only if $\tilde{\alpha}$ is a loop in E at e_0. We have $\tilde{\alpha}(0) = e_0$ and $\tilde{\alpha}(1) = \alpha^{\#}$, so $\tilde{\alpha}$ is a loop if and only if $\alpha^{\#} = e_0 = (e_{b_0})^{\#}$. But this is true if and only if $\alpha \sim e_{b_0}$, which is true because the endpoints agree and $[\alpha] = [\alpha * \overline{e_{b_0}}] \in H$.

The uniqueness of (E, e_0) (up to equivalence) has already been proved. □

To summarize the proof, we have the following bijections.

- We assume B is path connected, locally path connected, and semilocally simply connected. Then there is a bijection from subgroups $H \subseteq \pi_1(B, b_0)$ with basepoint preserving equivalence classes of covering spaces $p\colon (E, e_0) \to (B, b_0)$ with E path connected. Here, (E, e_0) maps to $p_*\pi_1(E, e_0)$.
- There is a bijection between conjugacy classes of $H \subseteq \pi_1(B, b_0)$ and equivalence classes of covering spaces $p\colon E \to B$ with E path connected. Here, E maps to $p_*\pi_1(E, e_0)$ where $e_0 \in p^{-1}(b_0)$ is any base point.

We call a covering space $p\colon E \to B$ with E path connected a *universal cover* of B if E is simply connected. Of course, $p_*\pi_1(E, e_0)$ is trivial, so p is unique up to equivalence by the second bijection above (if it exists).

Corollary. If B is path connected and locally path connected, then B has a universal covering if and only if B is semilocally simply connected.

Proof. If B has a universal covering, then we already showed this. Conversely, the classification theorem applied to $H = 0$ gives a covering space $p\colon E \to B$ with $p(e_0) = b_0$, E path connected, and $p_*\pi_1(E, e_0) = 0$. But p_* is injective, so $\pi_1(E, e_0)$ is trivial. □

Remark. Every n-manifold M that is path connected has a universal covering.

Example. The universal cover of $\mathbb{S}^1$ is the usual covering space $p\colon \mathbb{R} \to \mathbb{S}^1$. ◇

Example. The universal cover of $\mathbb{T}^n$ is the usual covering space $p\colon \mathbb{R}^n \to \mathbb{T}^n$. ◇

Example. For $n \geq 2$, the identity $\mathbb{S}^n \to \mathbb{S}^n$ is the universal cover. ◇

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