NYU Math Society Talk: Self-Similar Singularity Formation in PDEs

Grayson Davis 10/01/2025

1. Transport Equations

Motivation: Linear transport equations.

$$\begin{cases} u_t + \underbrace{c}_{\text{speed}} u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \qquad (T)$$

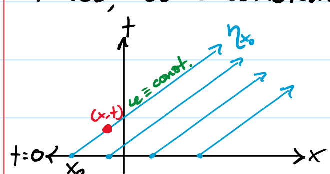
This PDE describes the propagation of the initial state to the right with speed c. To see this, let us solve (T).

We can rewrite (T) as $(u_t, u_x) \cdot (1, c) = 0$, or

directional derivative

$$\nabla_{(t,x)} u \cdot (1,c) = 0.$$

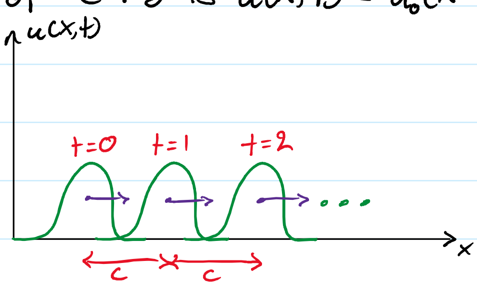
Thus, u is constant on all lines with slope $= c$.



We have $u(t, x(t)) \circ \eta_{x_0} = u_0(x_0)$.

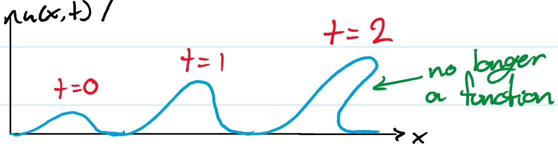
characteristic line starting at x_0

Trace back line to get $x_0 = x - ct$, so the general solution of (T) is $u(x,t) = u_0(x-ct)$.



This is great, but there are numerous phenomena where the exact structure of the initial state is not the same at every later time.

e.g. Waves



2. Burgers' Equation

We will study a non-linear transport equation where the speed is proportional to the solution itself.

This is the Burgers' equation:

$$\begin{cases} u_t + u u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \qquad (B)$$

We can solve (B) as we did before.

Let $z(t) = (u \circ \eta_{x_0})(t) = u(t, x(t))$, so that

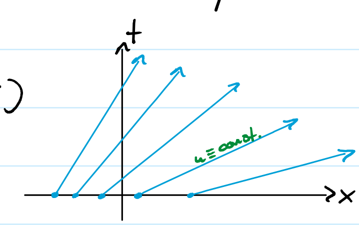
$$\underbrace{\frac{dx}{dt}}_{=\eta_0'(t)} = z(t), \quad \frac{dz}{dt} = u_t + \frac{dx}{dt} u_x = u_t + u u_x = 0.$$

Thus, $z \equiv \text{const}$, or $z(t) = u_0(x_0)$ for all $t > 0$.

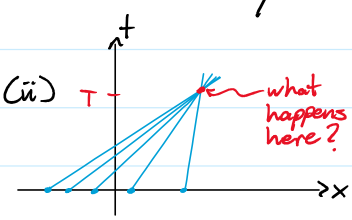
This means the characteristics are just lines $\left(\eta_0' = u_0(x_0)\right)$.

However, the slope of each line can vary.

e.g. (i)



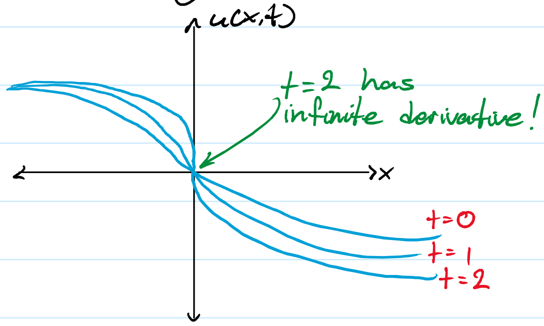
(ii)



In (ii), the solution u becomes discontinuous at some finite time $t = T$ - on one characteristic line, $u \circ \eta_{x_0} = C_1$, and on another $u \circ \eta_{y_0} = C_2$. If u is not constant, then at $t = T$ u becomes multivalued!

We call this type of singularity a shock.

Another view:



This illustrates how Burgers' connects to the breaking wave example from before.

Let us see this more mathematically.

We can compute $\frac{d}{dt} u_x(x(t),t) = u_{xt} + u u_{xx}$,

and $u_t + u u_x = 0$ gives $u_{tx} + u u_{xx} + u_x^2 = 0$.

Thus, $\frac{d}{dt}(u_x \circ \eta_{x_0}) = -(u_x \circ \eta_{x_0})^2$.

This is an ODE with solution

$u_x \circ \eta_{x_0}(t) = \frac{\alpha}{1+\alpha t}$, where $\alpha = u_0'(x_0)$.

In particular, u_x blows-up at the time $T = -\frac{1}{\max_{x_0 \in \mathbb{R}} u_0'(x_0)}$.

This illustrates the vertical derivative in the previous picture.

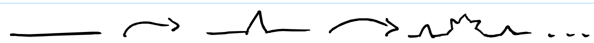
3. Self-Similar Shock Description (Reference: Eggers & Fontelos 2015)

We now want to provide a description of the solution at the time and location of the shock.

The key idea is to look for self-similar solutions.

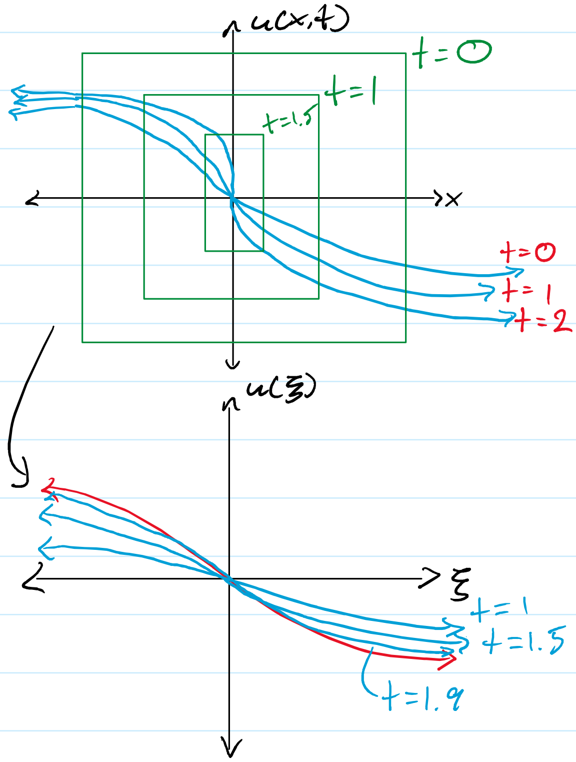
Self-similarity: when an object is exactly or approximately similar to a part of itself, independent of the scale.

e.g. fractals exhibit self-similarity.



What does self-similar shock formation mean?

Consider the picture before:



If we rescale the box in the first picture and look at the solution inside the box (in self-similar coordinates), we no longer see the infinite derivative but instead see a smooth function.

How can we describe this mathematically?

We look for solutions of Burgers' of the form

$$u(x,t) = (T-t)^{\alpha}\,U\!\left(\frac{x-x_0}{(T-t)^{\beta}}\right), \quad (A)$$

where (x_0, t_0) is position of the first shock, and $\alpha, \beta \geq 0$ are the self-similar scaling parameters (how the box in the above picture is scaled). Write $\xi := \frac{x-x_0}{(T-t)^{\beta}}$.

Observation 1: $\beta = 1+\alpha$.

To see this, we plug (A) into (B) $u_t + uu_x = 0$:

$$-\alpha(T-t)^{\alpha-1}U_{\xi} + (T-t)^{\alpha}(T-t)^{-\beta-1}(x-x_0)U_{\xi} + (T-t)^{\alpha}(T-t)^{-\beta}U_{\xi}(T-t)^{\alpha}U = 0.$$

$$= -\alpha(T-t)^{\alpha-1}U + \beta(T-t)^{\alpha-1}\,\xi\,U_{\xi} + (T-t)^{2\alpha-\beta}UU_{\xi} = 0.$$

We need each term to scale the same in t as $t \to T$, so

$\alpha - 1 = 2\alpha - \beta$, or $\beta = 1+\alpha$.

Thus, the self-similar Burgers' equation is the ODE:

$$-\alpha U + (1+\alpha)\xi U_\xi + UU_\xi = 0 \quad \text{(SB)}.$$

We can solve the ODE implicitly by the equation

$\xi = -U - CU^{1+\frac{1}{\alpha}}$, where C is an integration constant, $\alpha \neq 0$.

Exercise: Derive this equation.

Remark: For $\alpha = 0$, we get the solution $U = -\xi$.

This solution doesn't match the asymptotics we need U to satisfy, so we will exclude this solution (I am omitting the details from our discussion).

Now, if U is to be smooth, we need that $1+\frac{1}{\alpha}$ is an integer. Moreover, for U to be globally-defined, this integer must be odd. We also need $C > 0$.

This can be thought of as requiring the function $\xi(U) = -U - CU^{1+\frac{1}{\alpha}}$ to be a smooth diffeomorphism of $\mathbb{R}$.

Thus, we find a sequence of solutions U_i, defined by the similarity exponents $\alpha_i = \frac{1}{2i+2}$, $i = 0, 1, 2, \ldots$

In summary, we have found a sequence of smooth solutions $\{U_i\}_{i=0}^{\infty}$ to (SB) with exponents α_i.

Question: Which of these solutions are actually observed?

That is, which of these is the curve in the picture above?

4. Stability

To answer these questions, we must analyze the stability of each profile U_i.

Let us introduce the self-similar time variable

$s = -\log(T-t)$, which moves the blow-up time to infinity.

Then (SB) becomes

$$U_s + (-\alpha U + (1+\alpha)\xi U_\xi + UU_\xi) = 0 \quad (D),$$

a dynamical system.

Observation: Each U_i is a fixed point of (D).

Thus, we can discuss the stability of each U_i.

Consider the perturbation $U(\xi, s) = \bar{U}(\xi) + \delta e^{\lambda s}\tilde{U}(\xi)$,

where $\delta > 0$ is a small parameter, λ is some to be determined number (called an eigenvalue), and $\tilde{U}$ some perturbation function.

If we plug this into (D) and consider that equation to leading order (i.e. ignore $O(\delta^2)$ terms since they are small), we get an eigenvalue problem $\mathcal{L}\tilde{U} = \lambda \tilde{U}$, where $\mathcal{L}$ is the linear operator

identity operator

$$\mathcal{L} = \left(\alpha_i - \frac{\partial \bar{U}_i}{\partial \xi}\right) \mathrm{Id} - \left((1+\alpha_i)\xi - \bar{U}_i\right)\frac{\partial}{\partial \xi}.$$

Using $\xi = -\bar{U}_i - C\bar{U}_i^{2i+3}$, one can solve $\mathcal{L}\tilde{U} = \lambda \tilde{U}$,

$$\tilde{U} = \frac{\bar{U}_i^{3+2i-2\lambda(i+1)}}{1+(2i+3)\bar{U}_i^{2i+2}}.$$

Again, if $\tilde{U}$ is to be a smooth perturbation, we need $3+2i-2\lambda(i+1)$ to be some integer, so

$$\lambda_j^{(i)} = \frac{2i+3-j}{2i+2}.$$

Remark: Since we are intested in the first instance at which a shock forms, we want profiles with $U''(0)=0$ (i.e. u' has a global minimum at $\xi = 0$, so the 2nd derivative vanishes).

This means that the amplitude of the $j=2$

perturbation must be equal to zero (easy to see using a Taylor series expansion).

After eliminating the invalid eigenvalues, the remaining eigenvalues for $i=0$ are $\lambda^{(0)} = \frac{3}{2}, 1, 0, -\frac{1}{2}, \ldots$.

We want all eigenvalues to be negative to conclude that the $i=0$ profile is stable. Despite the presence of the eigenvalues $3/2 = 1+\alpha_0$, 1, and 0, we claim that U_0 is indeed stable.

To see why, we need to understand where the unstable directions from $3/2, 1$ come from.

The neutral direction $\lambda = 0$ is due to the constant C, which parametrizes a continuous family of solutions. This neutral direction can be removed by fixing C.

The other two unstable directions from $\lambda = 3/2, 1$ come from the idea that we can move the location and time of the shock, respectively:

if we move the location of the shock, the solution now blows-up at another point, so the perturbed profile converges to the profile corresponding to the new location;

the solution is pushed away from the similarity solution.

These two unstable directions can again be removed by fixing the location and time of blow-up (i.e. the unstable directions correspond to symmetries of Burgers' equation).

Overall, the $i=0$ profile $\bar{U}_0$ is stable.

For $i=1$, we have $\lambda = 5/4, 1, 1/2, 1/4, \ldots$, and for $i>1$ there are even more positive eigenvalues.

Thus, all profiles are unstable for $i \geq 1$:

the curve in the previous picture is $\bar{U}_0$.

In summary, we observe the solution

$u(x,t) = (T-t)^{1/2} \bar{U}_0\left(\frac{x-x_0}{(T-t)^{3/2}}\right)$, where $\bar{U}_0$

solves the equation $\xi + \bar{U}_0 + C\bar{U}_0^3 = 0$

solves the equation $\xi + \bar{U}_0 + C\bar{U}_0^3 = 0$.

5. Bonus: The Heat Equation

Consider the heat (diffusion) equation

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d \end{cases} \qquad (H)$$

What does this equation represent?

If u represents a heat distribution, $u_t = \Delta u$ says that the time-evolution of temperature is related to how smooth the distribution u is.

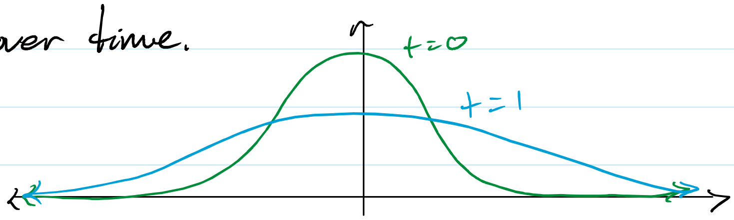
Suppose we take a region where u is highly oscillatory, i.e. over a small region of space, u has large variations in temperature. Then Δu is large, so $u_t = \Delta u$ is as well; over a short time the change in temperature is large.

As the very hot regions cool down and the very cold regions warm up, Δu becomes smaller, and hence so does u_t.

The large oscillations get "smoothed out" or "killed" over time.

Large oscillations are killed quickly, but small oscillations smoothen

Large oscillations are killed quickly, but small oscillations smoothen out over time.



If we start with a Gaussian, we expect it to get stretched out over time. We can also model this idea using self-similarity.

We look for a self-similar solution $u(x,t) = t^{\alpha} U(\frac{x}{t^{\beta}})$ to (H).

Moreover, we will impose radial symmetry on U,

$$U(\tfrac{x}{t^{\beta}}) = U(r) \quad , \quad r = |x|/t^{\beta}.$$

Then, (H) reduces to

$$t^{\alpha-1}[\alpha U(\xi) - \beta \xi U'(\xi)] - t^{\alpha-2\beta}[U''(\xi) + \frac{d-1}{\xi} U'(\xi)] = 0,$$

where we are using the radial Laplacian $\Delta f(r) = f_{rr} + \frac{d-1}{r} f_r$.

So the two terms scale the same, we require $\alpha - 1 = \alpha - 2\beta$, or $\beta = 1/2$.

The similarity equation for (H) hence reduces to

$$-\alpha U + (\tfrac{1}{2} r + \tfrac{d+1}{r}) U' + U'' = 0 \text{ (where } U' = \tfrac{d}{dr} U).$$

Choose $\alpha = -\frac{d}{2}$, so that this reduces to

$$[r^{d-1} U' + \tfrac{1}{2} r^d U]' = 0 \qquad r^{d-1} U' + \tfrac{1}{2} r^d U = A$$

$$\left[r^{d-1}U' + \frac{1}{2}r^d U\right]' = 0, \quad \text{or} \quad r^{d-1}U' + \frac{1}{2}r^d U = A,$$

for some constant A.

If we add on the decay assumptions

$$\lim_{r\to\infty} U(r) = 0, \quad \lim_{r\to\infty} U'(r) = 0,$$

then we conclude $A = 0$.

Solving the ODE, we find $U = Be^{-\frac{r^2}{4}}$ for some constant B. Thus, $u(x,t) = \frac{B}{t^{d/2}} e^{-\frac{|x|^2}{4t}}$.

This leads to the definition of the fundamental solution of the heat equation:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R},\ t > 0 \\ 0 & x \in \mathbb{R},\ t < 0 \end{cases}.$$