Blow-up Methods for Schauder Estimates: From Elliptic to Kinetic Equations

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1. Elliptic PDE

1.1. **Background.** The purpose of these notes is to illustrate the blow-up method in proving Schauder estimates for non-divergence form elliptic operators and the generalization of these methods to more complicated problems in PDE.

For the first part of these notes, we are interested in the equation

$$\operatorname{tr}\big(A(x)D^2u(x)\big) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}u(x) = f(x) \quad \text{in } B_r = B(0,r) \subseteq \mathbb{R}^d, \tag{1.1}$$

where $A(x) = (a_{ij}(x))_{ij}$ is a *uniformly elliptic matrix*, in the sense that there exist $\lambda, \Lambda > 0$ so that

$$\lambda \mathrm{Id} \le A(x) \le \Lambda \mathrm{Id} \quad \text{in } B_r.$$

That is, the differences $A - \lambda\mathrm{Id}$ and $\Lambda\mathrm{Id} - A$ are positive semi-definite. We prove the following theorem:

Theorem 1.1 (Schauder estimates for non-divergence form operators)**.** *Let $\alpha \in (0,1)$, and suppose $f, a_{ij} \in C^{0,\alpha}(B_1)$, and that $A = (a_{ij})_{ij}$ is uniformly elliptic in B_1 with ellipticity constants $\lambda, \Lambda > 0$. If $u \in C^{2,\alpha}(B_1)$ solves (1.1), then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)}), \tag{1.2}$$

where the constant $C > 0$ depends only on $\alpha, d, \lambda, \Lambda, \|a_{ij}\|_{C^{0,\alpha}(B_1)}$.

While numerous proofs of this theorem exist, we will prove it the robust blow-up method, introduced in [4]. Before illustrating this method, we briefly discuss the intuition behind Schauder estimates.

At its core, Schauder theory is a *perturbative theory*, as we treat the variables-coefficients equation as a small perturbation of the constant-coefficients (i.e. Laplacian) case at sufficiently small scales. Because the coefficients $a_{ij}(x)$ are Hölder continuous, locally, they oscillate slow enough so that the "rough" variables coefficients equation is well-approximated by the "smooth" constant-coefficients equation. In particular, the regularity from the constant-coefficients equation lifts to regularity for the full variable-coefficients case. We can see this heuristically as follows. For simplicity of notation, we use summation notation for the remainder of this document. Let us rewrite (1.1) as

$$a_{ij}(0)D^2u = f + [a_{ij}(0) - a_{ij}(x)]D^2u.$$

Then, since the a_{ij} are Hölder continuous, locally, we have

$$\big|a_{ij(0)} - a_{ij(x)}\big| \le Cr^\alpha,$$

so at scale r, (1.1) looks like

$$a_{ij}(0)D^2u = f + O(r^\alpha).$$

But r^α is small, so we can write $u = v+w$, where v comes from the constant-coefficients equation (so v has better regularity) and w is a small corrector. In particular, u will inherit the good regularity of v. This argument outlines the key ideas behind the "freezing the coefficients" proof of Schauder estimate.

It is important to note that we are specifically using that the coefficients are Hölder continuous and not merely continuous. If the a_{ij} were only continuous, there is no specified decay rate r^α, so one cannot expect to gain two whole derivatives. However, one can show (e.g. using the blow-up method of these notes) that for $f \in L^\infty(B_1)$ and $a_{ij} \in C^0(B_1)$, the solution u has regularity $C^{1,1-\epsilon}(B_{1/2})$ for any $\epsilon > 0$.

We instead present a dynamic argument to prove Theorem 1.1, where we "zoom in" until the geometry stabilizes to the constant-coefficients case. The idea is to assume the static "freezing" fails, which provides a sequence of solutions that become increasingly more oscillatory relative to f. We can then rescale each element of this sequence and check that they satisfy Poisson's equation in balls that increase in size as we go further down the sequence. By compactness, we can extract a subsequence that converges to a bounded solution of Laplace's equation, which will provide a contradiction. The main idea is that, the geometry looks flat as we dynamically "zoom in" to smaller scales, so that the solution gains regularity.

1.2. **Proof of Theorem 1.1.** We roughly follow the exposition of [1, Chapter 2].

We will instead prove the following proposition, and then the theorem can be shown after a little more work (which we will omit in these notes).

Proposition 1.1. *Under the assumptions of Theorem 1.1, for any $\delta > 0$, it follows that*

$$[D^2u]_{C^{0,\alpha}(B_{1/2})} \leq \delta[D^2u]_{C^{0,\alpha}(B_1)} + C_\delta(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)}), \tag{1.3}$$

where C_δ depends on $\delta, \alpha, d, \lambda, \Lambda, \|a_{ij}\|_{C^{0,\alpha}(B_1)}$.

The estimate (1.3) is almost the Schauder estimate (1.2). If the constant C_δ were to remain bounded in the limit $\delta \to 0$, then we would be done after applying an interpolation inequality. Moreover, if the Hölder norm on the right-hand side was in $B_{1/2}$ instead of B_1, we would also be done.

Proof of Proposition 1.1. After applying the interpolation inequality

$$\left\|D^2u\right\|_{L^\infty(B_1)} \leq \epsilon[D^2u]_{C^{0,\alpha}(B_1)} + C_\epsilon\|u\|_{L^\infty(B_1)},$$

we only have to prove that, for any $\delta > 0$ sufficiently small,

$$[D^2u]_{C^{0,\alpha}(B_{1/2})} \leq \delta[D^2u]_{C^{0,\alpha}(B_1)} + C_\delta(\left\|D^2u\right\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(B_1)}).$$

Suppose instead this inequality does not hold. Then there are sequences $\{u_k\}_k$, $\{f_k\}_k$, and $\{a_{ij}^{(k)}\}_k$ such that

$$a_{ij}^{(k)}(x)D^2u_k(x) = f_k(x) \quad \text{in } B_1$$

and for some fixed small $\delta_0 > 0$, we have

$$[D^2u_k]_{C^{0,\alpha}(B_{1/2})} > \delta_0[D^2u]_{C^{0,\alpha}(B_1)} + k(\left\|D^2u\right\|_{L^\infty(B_1)} + [f_k]_{C^{0,\alpha}(B_1)}). \tag{1.4}$$

First, we choose $x_k, y_k \in B_{1/2}$ such that

$$\frac{1}{2}[D^2u_k]_{C^{0,\alpha}(B_{1/2})} \leq \frac{\left|D^2u_k(x_k) - D^2u_k(y_k)\right|}{|x_k - y_k|^\alpha}.$$

Define the sequence of scales $\rho_k = |x_k - y_k|^\alpha$. We claim that $\rho_k \to 0$ as $k \to \infty$. Indeed,

$$\frac{1}{2}[D^2 u_k]_{C^{0,\alpha}(B_{1/2})} \leq \frac{\left\|D^2 u_k\right\|_{L^\infty(B_1)}}{\rho_k^\alpha} \leq \frac{[D^2 u_k]_{C^{0,\alpha}(B_{1/2})}}{k\rho_k^\alpha},$$

where we have used the assumption (1.4). In particular,

$$\rho_k \leq \frac{1}{2}k^{-\frac{1}{\alpha}} \to 0$$

as $k \to \infty$. The idea now is to rescale the u_k to get a sequence $\{\tilde{u}_k\}_k$ which solves some elliptic equation on balls of increasing size $B_{1/(2\rho_k)}$, and use compactness to extract some convergent subsequence which solves a constant-coefficients equation (which is equivalent to the Laplacian). We define

$$\tilde{u}_k(x) \coloneqq \frac{u_k(x_k + \rho_k x) - p_k(x)}{\rho_k^{2+\alpha}[D^2 u_k]_{C^{0,\alpha}(B_1)}},$$

$$\tilde{f}_k(x) \coloneqq \frac{f_k(x_k + \rho_k x) - f_k(x_k)}{\rho_k^\alpha [D^2 u_k]_{C^{0,\alpha}(B_1)}},$$

$$\tilde{a}_{ij}^{(k)}(x) \coloneqq a_{ij}^{(k)}(x_k + \rho_k x),$$

where p_k is the 2nd order Taylor expansion of $u_k(x_k + \rho_k x)$ at $x = 0$,

$$p_k(z) \coloneqq u_k(x_k) + \rho_k \sum_{i=1}^d \partial_i u_k(x_k) z_i + \frac{1}{2}\rho_k^2 \sum_{i,j=1}^d \partial_{ij} u_k(x_k) z_i z_j.$$

In particular, this choice of p_k imposes the condition

$$\tilde{u}_k(0) = |\nabla \tilde{u}_k(0)| = \left|D^2 \tilde{u}_k(0)\right| = 0. \tag{1.5}$$

Let us first show that $D^2\tilde{u}_k$ is bounded in $C^{0,\alpha}(B_{1/(2\rho_k)})$. Observe that $D^2 p_k = \rho_k^2 D^2 u_k(x_k)$, so that

$$D^2 \tilde{u}_k = \frac{\rho_k^2 D^2 u_k(x_k + \rho_k x) - \rho_k^2 D^2 u_k(x_k)}{\rho_k^{2+\alpha}[D^2 u_k]_{C^{0,\alpha}(B_1)}} = \frac{D^2 u_k(x_k + \rho_k x) - D^2 u_k(x_k)}{\rho_k^\alpha [D^2 u_k]_{C^{0,\alpha}(B_1)}},$$

from which we compute

$$\frac{\left|D^2\tilde{u}_k(x) - D^2\tilde{u}_k(y)\right|}{|x-y|^\alpha} = \frac{1}{[D^2 u_k]_{C^{0,\alpha}(B_1)}} \cdot \frac{\left|D^2 u_k(x_k + \rho_k y_k) - D^2 u_k(x_k + \rho_k y)\right|}{|\rho_k x - \rho_k y|^\alpha}.$$

Because $|\rho_k x| < 1/2$ and $x_k \in B_{1/2}$, we have $X = x_k + \rho_k x \in B_1$, and similar for $Y = x_k + \rho_k y$. Thus, we find the bound

$$[D^2 \tilde{u}_k]_{C^{0,\alpha}(B_{1/(2\rho_k)}} \leq 1. \tag{1.6}$$

Moreover, using (1.4), it follows that

$$\left|D^2\tilde{u}_k(\xi_k)\right| > \frac{\delta_0}{2},$$

where $\xi_k = (y_k - x_k)/2 \in \mathbb{S}^{d-1}$. Indeed,

$$\left|D^2\tilde{u}_k(\xi_k)\right| = \frac{1}{[D^2 u_k]_{C^{0,\alpha}(B_1)}} \cdot \frac{\left|D^2 u_k(x_k) - D^2 u_k(y_k)\right|}{\rho_k^\alpha} \geq \frac{1}{2} \cdot \frac{[D^2 u_k]_{C^{0,\alpha}(B_{1/2})}}{[D^2 u_k]_{C^{0,\alpha}(B_1)}} > \frac{\delta_0}{2},$$

by our choice of x_k, y_k and (1.4).

In particular, (1.5) and (1.6), we have that $\tilde{u}_k$ is uniformly bounded in compact subsets of $\mathbb{R}^d$ and bounded in the $C^{2,\alpha}$ norm. Thus, by Arzelà-Ascoli, $\tilde{u}_k$ converges (up to a subsequence) in the

C^2 norm to some $C^{2,\alpha}$ function $\tilde{u}$ on compact subsets of $\mathbb{R}^d$. Moreover, $\xi_k \to \xi \in \mathbb{S}^{d-1}$, up to a subsequence, and $\tilde{u}$ satisfies the same properties as the $\tilde{u}_k$ on the whole space:

$$\tilde{u} = |\nabla \tilde{u}| = \left|D^2\tilde{u}\right| = 0, \quad [D^2\tilde{u}]_{C^{0,\alpha}(\mathbb{R}^d)} \leq 1, \quad \left|D^2\tilde{u}(\xi)\right| > \frac{\delta_0}{2}.$$

Let us now show that the $\tilde{f}_k$ and $\tilde{a}_{ij}^{(k)}$ converge in compact subsets of $\mathbb{R}^d$ to zero and a constant, respectively. First, for any $R \geq 1$, we compute

$$\left\|\tilde{f}_k\right\|_{L^\infty(B_R)} \leq \frac{(\rho_k R)^\alpha [f_k]_{C^{0,\alpha}(B_1)}}{\rho_k^\alpha [D^2u_k]_{C^{0,\alpha}(B_1)}} \leq \frac{R^\alpha [D^2u_k]_{C^{0,\alpha}(B_{1/2})}}{k[D^2u_k]_{C^{0,\alpha}(B_1)}} \leq \frac{R^\alpha}{k},$$

so that $\tilde{f}_k \to 0$ on compact subsets of $\mathbb{R}^d$. Moreover,

$$[\tilde{a}_{ij}^{(k)}]_{C^{0,\alpha}(B_{1/(2\rho_k)}} \leq \rho_k^\alpha [a_{ij}^{(k)}]_{C^{0,\alpha}(B_1)},$$

so that $\tilde{a}_{ij}^{(k)}$ converges uniformly in compact subsets of $\mathbb{R}^d$ (up to a subsequence) to some constant $\tilde{a}_{ij}$.

Now, $\tilde{u}_k$ solves the equation

$$\tilde{a}_{ij}^{(k)}\partial_{ij}\tilde{u}_k = \tilde{f}_k - \frac{\left(a_{ij}^{(k)}(x_k + \rho_k x) - a_{ij}^{(k)}(x_k)\right)\partial_{ij}u_k(x_k)}{\rho_k^\alpha [D^2u]_{C^{0,\alpha}(B_1)}},$$

where we recall that the summation is taken over i, j. We want to take the limit of this equation as $k \to \infty$. Observe that

$$\left|\tilde{a}_{ij}^{(k)}\partial_{ij}\tilde{u}_k - \tilde{f}_k\right| \leq \frac{|x|^\alpha \rho_k^\alpha [\tilde{a}_{ij}^{(k)}]_{C^{0,\alpha}(B_1)} \|\partial_{ij}u_k\|_{L^\infty(B_1)}}{\rho_k^\alpha [D^2u_k]_{C^{0,\alpha}(B_1)}} \leq C|x|^\alpha \frac{\left\|D^2u_k\right\|_{L^\infty(B_1)}}{[D^2u_k]_{C^{0,\alpha}(B_{1/2})}}.$$

Hence, we can use the contradiction assumption (1.4) to say that, given $x \in B_\sigma$ for some fixed $\sigma \in (0, \infty)$, and for k large enough,

$$\left|\tilde{a}_{ij}^{(k)}\partial_{ij}\tilde{u}_k - \tilde{f}_k\right| \leq C_\sigma k^{-1}.$$

Taking $k \to \infty$, we deduce the constant coefficients equation

$$\tilde{a}_{ij}\partial_{ij}\tilde{u} = 0 \quad \text{in } \mathbb{R}^d,$$

which, up to an affine change-of-coordinates, is equivalent to $\Delta\tilde{u} = 0$. However, $[D^2\tilde{u}]_{C^{0,\alpha}(\mathbb{R}^d)} \leq 1$, so $D^2\tilde{u}$ has sublinear growth at infinity, and hence $\tilde{u}$ is a quadratic polynomial. However, we have fixed the values of $\tilde{u}$, $\nabla\tilde{u}$, and $D^2\tilde{u}$ at the origin to be zero, so in fact $\tilde{u} \equiv 0$. But $\left|D^2\tilde{u}(\xi)\right| > \delta_0/2$, a contradiction. □

For any $\delta > 0$, we have shown the estimate

$$[D^2u]_{C^{0,\alpha}(B_{1/2})} \leq \delta[D^2u]_{C^{0,\alpha}(B_1)} + C_\delta(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)}), \tag{1.7}$$

where C_δ depends on $\delta, \alpha, d, \lambda, \Lambda, \|a_{ij}\|_{C^{0,\alpha}(B_1)}$.

The idea in concluding the proof is to define another seminorm which measures how the $C^{2,\alpha}$ norm blows up near ∂B_1. This seminorm is defined by

$$[D^2u]^*_{\alpha,B_1} := \sup_{B_\rho(x_0)\subseteq B_1} \rho^{2+\alpha}[D^2u]_{C^{0,\alpha}(B_{\rho/2}(x_0))}.$$

We omit the details of the proof (see [1] for a complete argument).

We remark that the same blow-up argument can be applied to prove a Schauder estimate for divergence-form elliptic operators.

2. Kinetic PDE

The remainder of these talk notes will provide a very brief overview of how the blow-up argument can be applied to broader PDE contexts by studying the example of kinetic integral equations.

2.1. **Kinetic Integral Equations.** In [2], Imbert & Silvestre apply the blow-up argument to prove Schauder estimates for kinetic integral equations of the form

$$f_t + v \cdot \nabla_x f = \int_{\mathbb{R}^d} (f' - f) K(t, x, v, v')\, dv' + c_0(t, x, v),$$

where $f = f(t, x, v)$, $f' = f(t, x, v')$, and K is some kernel that is elliptic and Hölder continuous (as defined in [2]). It is important to note that, despite the right-hand-side having no diffusion in x, the transport term $v \cdot \nabla_x f$ provides regularization in x. This effect is easily seen via the Galilean left-invariance of (2.1): define

$$(t_1, x_1, v_1) \circ (t_2, x_2, v_2) = (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2),$$

then if $f(z)$ solves (2.1), so does $f_0(z) = f(z_0 \circ z)$ (with a translated right-hand side and kernel).

The blow-up argument can be adapted to this case to prove an estimate of the form

$$\|f\|_{C_l^{2s+\alpha}(Q_{1/2})} \leq C(\|f\|_{C_l^{\gamma}([-1,0]\times B_r \times \mathbb{R}^d)} + \|c\|_{C_l^{\alpha}(Q_1)}). \tag{2.1}$$

We will not go into the full details and definitions of everything here, but let us quickly comment on some of the notation.

First, the parameter s represents the diffusivity of the kernel K; the larger s is, the more singular K becomes. The parameter γ is any number satisfying $0 < \gamma < \min(1, 2s)$, and $\alpha = (2s/(1+2s))\gamma$. The constant C depends on the dimension, s, and the ellipticity and Hölder constants associated with the kernel K. The domain Q_r is the kinetic cylinder $Q_r = (-r^{2s}, 0] \times B_{r^{1+2s}} \times B_r$. The Hölder norms all refer to regularity in the v variable, with the appropriate regularity in t and x following from the invariance of the equation. The subscript l in the space C_l^{β} indicates that the Hölder spaces are defined using a distance that is left-invariant with respect to the Lie group structure.

The proof of (2.1) uses the same ingredients as the blow-up argument presented for the elliptic case. The authors prove a Liouville-type theorem that provides conditions for solutions to (2.1) to be a *kinetic* polynomial of degree $\alpha + 2s$. Proceeding by contradiction, the authors utilize compactness via Arzelà-Ascoli and then apply the Liouville theorem to conclude that (2.1) holds. While the general idea remains the same, much of the work is redefining the spaces and reproving the relevant estimates to define the correct setting in which they can apply the blow-up argument.

2.2. **Applications to Boltzmann Equation.** The Schauder estimate (2.1) can be applied to prove global regularity estimates for the Boltzmann equation. We will give a very brief overview of how this works, only to emphasize the usefulness of Schauder estimates in analyzing complicated non-linear PDE.

Recall that the spatially inhomogenous Boltzmann equation on $(0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ if of the form

$$f_t + v \cdot \nabla_x f = Q(f, f),$$

where Q is the Boltzmann collision operator

$$Q(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(v'_*) f(v') - f(v_*) f(v)) B(|v - v_*|, \cos(\theta))\, dv_* d\sigma.$$

Here,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|2}{\sigma}, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \cos(\theta) = \frac{v-v_*}{|v-v_*|}\cdot\sigma.$$

Physically, v' and v'_* are the post-collisional velocities of two colliding particles, and θ measures the deviation between v and v'. We will take the non-cutoff collisional kernel

$$B(r,\cos(\theta)) = r^\gamma b(\cos(\theta)),$$

where $b(\cos(\theta)) \approx |\sin(\theta/2)|^{-(d-1)-2s}$, with $\gamma > -d$ and $s \in (0,1)$. We also define the hydrodynamic quantities

$$\begin{aligned} M(t,x) &\coloneqq \int_{\mathbb{R}^d} f(t,x,v)\ dv \\ E(t,x) &\coloneqq \int_{\mathbb{R}^d} f(t,x,v)|v|^2\ dv \\ H(t,x) &\coloneqq \int_{\mathbb{R}^d} f(t,x,v)\log(f(t,x,v))\ dv, \end{aligned}$$

which are the mass, energy, and entropy densities, respectively. Under the assumptions that

$$0 < m_0 \le M(t,x) \le M_0, \quad E(t,x) \le E_0, \quad H(t,x) \le H_0$$

uniformly in t and x, Imbert & Silvestre [3] show that for any $k \in \mathbb{N}^{1+2d}$, $\tau > 0$, and $q > 0$,

$$\left|(1+|v|)^q D^k f\right|_{L^\infty([\tau,T)\times\mathbb{R}^d\times\mathbb{R}^d)} \le C(k,q,\tau), \tag{2.2}$$

where D^k is any arbitrary derivative of f of any order in t, x, and/or v. The constant C depends on $k, q, \tau, m_0, M_0, E_0, H_0, s, \gamma, d$ if $\gamma > 0$. For $\gamma \le 0$, the constant depends also on the pointwise decay of the initial data.

Part of the proof involves a change of variables that turns the local Schauder estimate (2.1) into a global Schauder estimate. Then, given this global estimate, they can formulate a bootstrapping mechanism by iterating the Schauder estimate.

The global Schauder estimate is applied to solutions to the *linear Boltzmann equation*,

$$g_t + v\cdot\nabla_x g = Q_1(f,g) + h,$$

where Q_1 is defined by splitting the Boltzmann collision operator as $Q = Q_1 + Q_2$ with

$$\begin{aligned} Q_1(f,f) &\coloneqq L_{K_f} f = \mathrm{PV}\int_{\mathbb{R}^d} (g(t,x,v') - g(t,x,v))K_f(t,x,v,v')\ dv', \\ Q_2(f,f) &\coloneqq c_b(f * |\cdot|^\gamma)f. \end{aligned}$$

Here, K_f is some kernel involving the non-cutoff collisional kernel. If set $f = g$ and $h = Q_2(f,f)$ in the linear equation, then the global Schauder estimates apply to the full (nonlinear) Boltzmann equation. The global Schauder estimates take the form

$$\|g\|_{C^{2s+\alpha'}([\tau,T]\times\mathbb{R}^d\times\mathbb{R}^d)} \le C\Big(\|g\|_{C^\alpha_{l,q+\kappa}(0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} + \|h\|_{C^{\alpha'}_{l,q+\kappa}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)}\Big), \tag{2.3}$$

where where q and $q+\kappa$ in the subscript of the Hölder spaces correspond to additional decay to the functions in C^α_l. The global estimate 2.3 is a consequence of the local Schauder estimate alongside a change-of-variables (and some other estimates in [3]), and is then a key ingredient in the proof of the global regularity estimates (2.2).

References

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