Honours Analysis II - Recitation 10

Wednesday, April 8, 2026 10:56 AM

Reminders: ▷ HW5 due Monday.

▷ MT2 on Wednesday.

Today: ▷ Midterm review!

<u>Change of Variables</u>:

Thm.: Let $U \subseteq \mathbb{R}^n$ be open and $g: U \to \mathbb{R}^n$ an injective, C^1 map such that $\det Dg(x) \neq 0$ for all $x \in U$. If $f: g(A) \to \mathbb{R}$ is integrable, then $\int_{g(A)} f = \int_A (f \circ g)|\det Dg|$.

1 (HW 4, 1(b)). Let $U \subseteq \mathbb{R}^n$ be open, $g \in C^1(U \to \mathbb{R}^n)$.

If $\det Dg(x_0) \neq 0$ at some $x_0 \in U$, show that $\lim_{r \downarrow 0} \frac{v(g(B_r(x_0)))}{v(B_r(x_0))} = |\det Dg(x_0)|$.

Sol'n.: Since $\det Dg(x_0) \neq 0$, by the inverse function theorem, there is a nbd. U of x_0 where g is a C^1 diffeomorphism. Let $r > 0$ be small enough so that $B_r(x_0) \subseteq U$.

Then by the change-of-variables formula,

$$v(g(B_r(x_0))) = \int_{g(B_r(x_0))} 1 \, dy = \int_{B_r(x_0)} |\det Dg(x)| \, dx.$$

Hence,

$$\lim_{r \downarrow 0} \frac{v(g(B_r(x_0)))}{v(B_r(x_0))} = \lim_{r \downarrow 0} \frac{1}{v(B_r(x_0))} \int_{B_r(x_0)} |\det Dg(x)| \, dx = |\det Dg(x_0)|,$$

since $|\det Dg(x)|$ is continuous. ∎

<u>Tensors</u>:

Def'n.: A multilinear map $T: V^k \to \mathbb{R}$ is called a k-tensor.

Given $S \in \mathcal{T}^k(V)$, $T \in \mathcal{T}^\ell(V)$, the tensor product is

$$\underbrace{(S \otimes T)}_{\in \mathcal{T}^{k+\ell}(V)}(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k)\, T(v_{k+1}, \ldots, v_{k+\ell}).$$

Thm.: If $\{v_1, \dots, v_n\}$ is a basis of V and $\{\varphi_1, \dots \varphi_n\}$ the dual basis, then $\{\varphi_{i_1} \otimes \dots \varphi_{i_k}\}_{i_j \in \{1, \dots, k\}}$ is a basis of $\mathcal{T}^k(V)$.

Hence, $\dim \mathcal{T}^k(V) = n^k$.

2 (HW 4, 2): Let $V = \mathbb{R}^2$ with dual basis φ_1, φ_2. Define $T \in \mathcal{T}^2(V)$ by

$T(v_1, v_2) = 3\varphi_1 \otimes \varphi_1 - \varphi_1 \otimes \varphi_2 + 2\varphi_2 \otimes \varphi_1 + 4\varphi_2 \otimes \varphi_2$.

(a) Write $T(v, w) = v^T A w$ for some matrix A.

(b) Define $S(v, w) = T(w, v)$. Express S in the basis and write S as a matrix.

(c) Prove in general that $T \in \mathcal{T}^2(V)$ is symmetric if and only if $A = A^T$, where A is the matrix for T.

Sol'n:

(a) Evaluate T at the basis elements $T(e_i, e_j)$ to get

$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, and can check

$$\begin{aligned} T(v, w) &= 3v_1w_1 - v_1w_2 + 2v_2w_1 + 4v_2w_2 \\ &= [v_1 \ v_2] \begin{bmatrix} 3w_1 - w_2 \\ 2w_1 + 4w_2 \end{bmatrix} \\ &= v^T A w. \end{aligned}$$

(b) We compute

$$\begin{aligned} S(v, w) = T(w, v) &= 3w_1v_1 - w_1v_2 + 2w_2v_1 + 4w_2v_2 \\ &= (3\varphi_1 \otimes \varphi_1 + 2\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 + 4\varphi_2 \otimes \varphi_2)(v, w). \end{aligned}$$

From this, $B = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} = A^T$.

(c) Write $T = a\varphi_1 \otimes \varphi_1 + b\varphi_1 \otimes \varphi_2 + c\varphi_2 \otimes \varphi_1 + d\varphi_2 \otimes \varphi_2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then T is symmetric $\iff T(e_i, e_j) = T(e_j, e_i) \iff b = c \iff A = A^T$. ▨

Then T is symmetric $\iff T(e_i, e_j) = T(e_j, e_i) \iff b = c \iff A = A^T$. ▨

Alternating tensors:

Def'n.: A tensor $\omega \in \mathcal{T}^k(V)$ is alternating if $\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$.

Given $T \in \mathcal{T}^k(V)$, define $\mathrm{Alt}(T) \in \Lambda^k(V)$ by $\mathrm{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

Exercise: Show that $\dim \Lambda^k(\mathbb{R}^k) = 1$.

3 (Munkres §27, 4). Let $T: V \to W$ be a linear transformation, and let $T^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ be the dual map, $(T^* S)(v_1, \dots, v_k) = S(T(v_1), \dots, T(v_k))$.

Show that if $\omega \in \Lambda^k(W)$, then $T^*\omega \in \Lambda^k(V)$.

Sol'n.: We compute

$$\begin{aligned}
(T^*\omega)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= \omega(T(v_1), \dots, T(v_i), \dots, T(v_j), \dots, T(v_k)) \\
&= -\omega(T(v_1), \dots, T(v_j), \dots, T(v_i), \dots, T(v_k)) \\
&= -(T^*\omega)(v_1, \dots, v_j, \dots, v_i, \dots, v_k).
\end{aligned}$$

▨

Wedge products:

Def'n.: Given $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, define

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta).$$

4. If $\omega_1, \dots, \omega_k$ are alternating tensors, then

$$\frac{(d_1 + \dots + d_k)!}{d_1! \cdots d_k!} \mathrm{Alt}(\omega_1 \otimes \dots \otimes \omega_k) = \omega_1 \wedge \dots \wedge \omega_k.$$

Sol'n.: We go by induction.

Assume the result holds for $k-1$. Then

$$\begin{aligned}
\omega_1 \wedge \dots \wedge \omega_k &= \left[\frac{(d_1 + \dots + d_{k-1})!}{d_1! \cdots d_{k-1}!} \mathrm{Alt}(\omega_1 \otimes \dots \otimes \omega_{k-1})\right] \wedge \omega_k \\
&=: \eta \wedge \omega_k
\end{aligned}$$

$$=: \eta \wedge \omega_k$$

But
$$\eta \wedge \omega = \frac{(d_\eta + d_k)!}{d_\eta!\, d_k!} \mathrm{Alt}(\eta \otimes \omega)$$
$$= \frac{(d_\eta + d_k)!}{\cancel{d_\eta!}\, d_k!} \left[\mathrm{Alt}\left(\frac{\cancel{d_\eta!}}{d_1! \cdots d_{k-1}!} \mathrm{Alt}(\omega_1 \otimes \cdots \otimes \omega_{k-1}) \otimes \omega_k\right)\right]$$
$$= \frac{(d_1 + \cdots + d_{k-1} + d_k)!}{d_1! \cdots d_{k-1}!\, d_k!} \mathrm{Alt}(\mathrm{Alt}(\omega_1 \otimes \cdots \otimes \omega_{k-1}) \otimes \omega_k)$$
$$= \frac{(d_1 + \cdots + d_k)!}{d_1! \cdots d_k!} \mathrm{Alt}(\omega_1 \otimes \cdots \otimes \omega_k),$$

where we have used that $\mathrm{Alt}(\mathrm{Alt}(S) \otimes T) = \mathrm{Alt}(S \otimes T)$ [exercise]. $\boxtimes$

<u>Differential forms</u>:

Def'n.: Let $\mathbb{R}^n_p = \{(p, v) : v \in \mathbb{R}^n\}$ be the tangent space at p. We call $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ a k-form.

Given the dual basis $\varphi_1, \ldots, \varphi_k$, we can write $\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k}(p) (\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k})$,

for some functions $\omega_{i_1, \ldots, i_k}$ on $\mathbb{R}^n$.

The standard basis is denoted $dx^1, \ldots, dx^n$:

Given $f: \mathbb{R}^n \to \mathbb{R}$ differentiable, define $df(p)(v_p) = Df(p)(v)$.

The pushforward of a differentiable function $f: \mathbb{R}^n \to \mathbb{R}^m$ is

$$f_*: \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}$$
$$v_p \longmapsto Df(p)(v) = df(p)(v_p).$$

The pullback $f^*: \Lambda^k(\mathbb{R}^m_{f(p)}) \longrightarrow \Lambda^k(\mathbb{R}^n_p)$ is defined by

$$(f^*\omega)(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k)).$$

Properties:
- $f^*(dx^i) = \sum_{j=1}^n \frac{\partial f^i}{\partial x_j} dx^j$ (i-th column of Df).
- $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$.
- $f^*(g\omega) = (g \circ f) f^*\omega$.
- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Given a k-form ω, we define the $(k+1)$-form $d\omega$ by

$$d\omega = \sum (d\omega_{i_1, \ldots, i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum \sum_{\alpha=1}^n (D_\alpha \omega_{i_1, \ldots, i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$d\omega = \sum_{i_1<\dots<i_k} (d\omega_{i_1\dots i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{i_1<\dots<i_k} \sum_{\alpha=1}^{n} (\partial_\alpha \omega_{i_1\dots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

A k-form is called closed if $d\omega = 0$, and exact if $\omega = d\eta$ for some $(k-1)$-form η.

Lemma (Poincaré): If Ω is a star-shaped open set (wrt 0), then every closed form on Ω is exact.

5 (Spivak 4-19): Let F be a vector field on $\mathbb{R}^3$. Define

$\omega_F^1 = F^1 dx + F^2 dy + F^3 dz$, $\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$.

(a) Prove that $df = \omega^1_{\text{grad } f}$, $d(\omega_F^1) = \omega^2_{\text{curl } F}$, $d(\omega_F^2) = (\text{div } F)\, dx \wedge dy \wedge dz$.

(b) Use (a) to prove that curl grad $f = 0$, div curl $F = 0$.

(c) If F is a vector field on a star-shaped open set Ω and curl $F = 0$, show that $F = \text{grad } f$ for some function $f: \Omega \to \mathbb{R}$.

Likewise, if div $F = 0$, then $F = \text{curl } G$ for some vector field G on Ω.

Sol'n: (a) I'll show $d(\omega_F^2) = \text{div } F\, dx \wedge dy \wedge dz$ and leave the other 2 as an exercise.

We compute $d(\omega_F^2) = (dF^1) \wedge dy \wedge dz + (dF^2) \wedge dz \wedge dx + (dF^3) \wedge dx \wedge dy$.

But $dF^i = \partial_1 F^i dx + \partial_2 F^i dy + \partial_3 F^i dz$, so

- $dF^1 \wedge dy \wedge dz = \partial_1 F^1 dx \wedge dy \wedge dz$,
- $dF^2 \wedge dz \wedge dx = \partial_2 F^2 dx \wedge dy \wedge dz$,
- $dF^3 \wedge dx \wedge dy = \partial_3 F^3 dx \wedge dy \wedge dz$.

Thus, $d(\omega_F^2) = \text{div } F\, dx \wedge dy \wedge dz$.

(b) We know $d^2 = 0$, so

$\omega^2_{\text{curl grad} f} = d(\omega^1_{\text{grad} f}) = d(df) = 0$.

Then the component vector field is 0, i.e. curl grad $f = 0$.

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We also have $d(\omega^2_{\text{curl} F}) = 0$, so that

$(\text{div curl } F)\, dx \wedge dy \wedge dz \equiv 0$, or $\text{div curl } F = 0$.

(c) This is now a immediate consequence of the Poincaré lemma